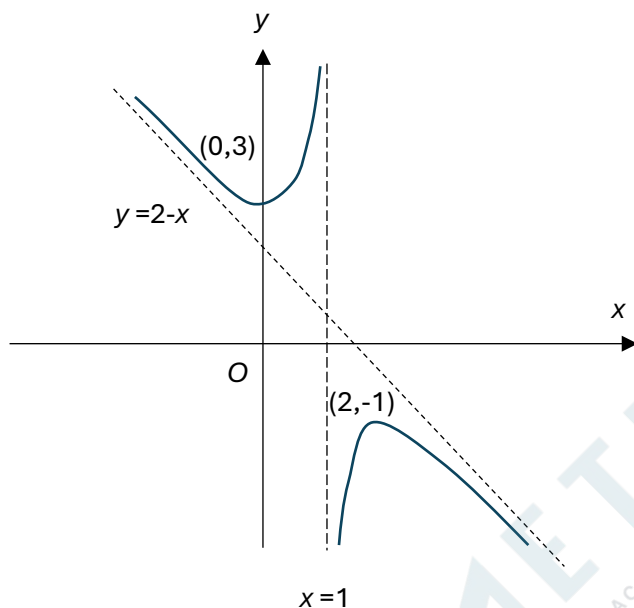


1

(a) $y = \frac{x^2 - 3x + 3}{1 - x} = 2 - x + \frac{1}{1 - x}$

Asymptotes: $y = 2 - x$, $x = 1$



(b) $\frac{x^2 - 3x + 3}{1 - x} = kx$

$$x^2 - 3x + 3 = kx(1 - x)$$

$$(1 + k)x^2 - (3 + k)x + 3 = 0$$

For two points of intersection, discriminant > 0 .

$$(3 + k)^2 - 4(1 + k)(3) > 0$$

$$9 + 6k + k^2 - 12 - 12k > 0$$

$$k^2 - 6k - 3 > 0$$

$$(k - 3)^2 - 12 > 0$$

$$k < 3 - 2\sqrt{3} \text{ or } k > 3 + 2\sqrt{3}$$

$$y = \frac{x^2 - 3x + 3}{1 - x} = 2 - x + \frac{1}{1 - x}$$

Consider the oblique asymptote of the curve C is $y = 2 - x$,

for two points of intersection between the curve and the line, the set of values of k is

$$\{k \in \mathbb{R} : k < 3 - 2\sqrt{3} \text{ or } k > 3 + 2\sqrt{3}, k \neq -1\}.$$

2

(i)

$$y = \frac{x^2 + x + 3}{x - 2}$$

$$yx - 2y = x^2 + x + 3$$

$$x^2 + (1 - y)x + 3 + 2y = 0$$

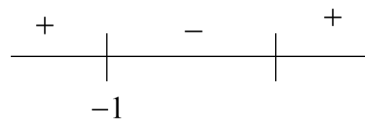
Consider Discriminant ≥ 0

$$(1 - y)^2 - 4(1)(3 + 2y) \geq 0$$

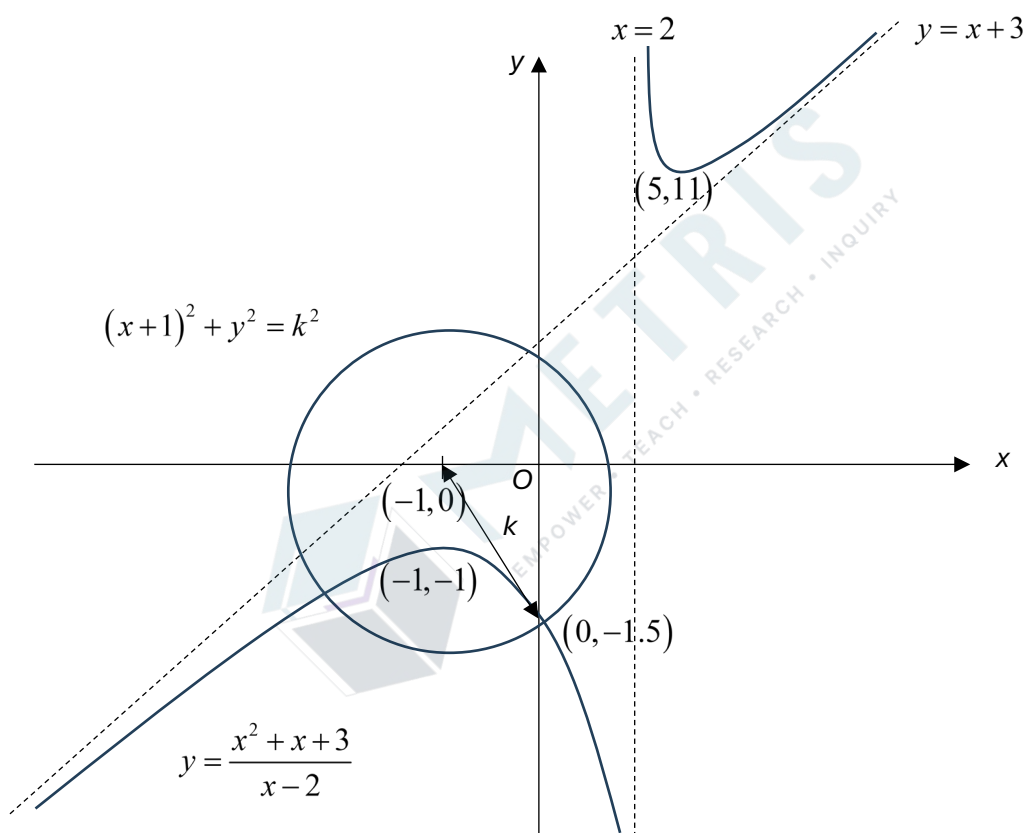
$$y^2 - 10y - 11 \geq 0$$

$$(y + 1)(y - 11) \geq 0$$

$$y \leq -1 \quad \text{or} \quad y \geq 11$$



(ii)



(iii)

$$(x+1)^2(x-2)^2 + (x^2 + x + 3)^2 = k^2(x-2)^2$$

$$(x+1)^2 + \left(\frac{x^2 + x + 3}{x-2} \right)^2 = k^2$$

$$(x+1)^2 + y^2 = k^2$$

For $(x+1)^2(x-2)^2 + (x^2 + x + 3)^2 = k^2(x-2)^2$ to have at least 1 positive real root,

Since $k > 0$, $k > \sqrt{(-1-0)^2 + (0-(-1.5))^2}$

$$k > \sqrt{3.25}$$

$$k > 1.803 \text{ (3 d.p.)}$$

3

(i) (ii)

$$y = \frac{-4x^2 - 6x - 11}{2x + 2}$$

$$y(2x + 2) = -4x^2 - 6x - 11$$

$$4x^2 + (6 + 2y)x + (11 + 2y) = 0$$

$$D < 0$$

$$(6 + 2y)^2 - 4(4)(11 + 2y) < 0$$

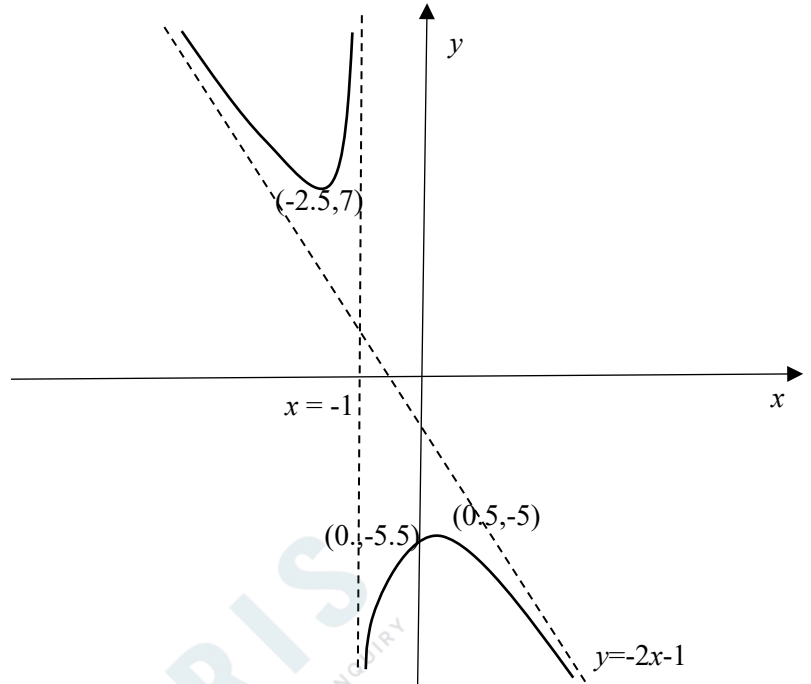
$$36 + 24y + 4y^2 - 176 - 32y < 0$$

$$4y^2 - 8y - 140 < 0$$

$$y^2 - 2y - 35 < 0$$

$$(y - 7)(y + 5) < 0$$

$$-5 < y < 7$$



(iii)

$$(x+1)^2 (2x+2)^2 - a^2 (-4x^2 - 8x - 13)^2 = a^2 (2x+2)^2$$

$$\frac{(x+1)^2 (2x+2)^2}{a^2 (2x+2)^2} - \frac{a^2 (-4x^2 - 8x - 13)^2}{a^2 (2x+2)^2} = \frac{a^2 (2x+2)^2}{a^2 (2x+2)^2}$$

$$\frac{(x+1)^2}{a^2} - \frac{(-4x^2 - 6x - 11 - 2x - 2)^2}{(2x+2)^2} = 1$$

$$\frac{(x+1)^2}{a^2} - \left(\frac{-4x^2 - 6x - 11}{2x+2} - 1 \right)^2 = 1$$

Hence consider the intersections of the curve C with the hyperbola $\frac{(x+1)^2}{a^2} - (y-1)^2 = 1$.

Centre of hyperbola: $(-1, 1)$

Equations of asymptotes of hyperbola: $y - 1 = \pm \frac{1}{a}(x + 1)$, which intersect at $(-1, 1)$.

Comparing $y - 1 = -\frac{1}{a}(x + 1)$ with the oblique asymptote of C ($y = -2x - 1$):

$$\therefore -\frac{1}{a} \geq -2 \Rightarrow a \geq \frac{1}{2} \text{ (for } a \text{ positive)}$$

(iv)

$$y = \frac{-4x^2 - 6x - 11}{2x + 2} = -2x - 1 - \frac{9}{2x + 2}$$

1. Reflection in the y -axis (or x -axis)

$$y = 2x + \frac{9}{2x} \rightarrow y = -2x - \frac{9}{2x}$$

2. Translation 1 unit in the negative x -direction

$$y = -2x - \frac{9}{2x} \rightarrow y = -2(x+1) - \frac{9}{2(x+1)}$$

3. Translation 1 unit in the positive y -direction

$$y = -2(x+1) - \frac{9}{2(x+1)} \rightarrow y = -2x - 1 - \frac{9}{2x + 2}$$

OR

1. Translation 1 unit in the positive x -direction

$$y = 2x + \frac{9}{2x} \rightarrow y = 2(x-1) + \frac{9}{2(x-1)}$$

2. Reflection in the y -axis

$$y = 2(x-1) + \frac{9}{2(x-1)} \rightarrow y = -2 - 2x - \frac{9}{2x + 2}$$

3. Translation 1 unit in the positive y -direction

$$y = -2 - 2x - \frac{9}{2x + 2} \rightarrow y = -1 - 2x - \frac{9}{2x + 2}$$



KETTRIS
EMPOWER • TEACH • RESEARCH • INQUIRY

(a) Method 1:Let $y = 0$

$$-x^2 + ax + b = 0$$

$$x^2 - ax - b = 0$$

$$\left(x - \frac{a}{2}\right)^2 - b - \left(\frac{a}{2}\right)^2 = 0$$

Since $(0,0)$ is a turning point, $a = b = 0$ **Method 2:**Since $(0,0)$ is a turning point,

$$0 = \frac{0+0+b}{4}$$

$$b = 0$$

$$y = \frac{-x^2 + ax + 0}{2x + 4} = -\frac{1}{2}x + \frac{a+2}{2} + \frac{-2a-4}{2x+4}$$

$$\begin{array}{r} -\frac{1}{2}x + \frac{a+2}{2} \\ 2x+4 \overline{) -x^2 + ax} \\ \underline{-x^2 - 2x} \\ (a+2)x \\ \underline{(a+2)x + 2a+4} \\ 2a+4 \end{array}$$

$$\frac{dy}{dx} = -\frac{1}{2} + \frac{4a+8}{(2x+4)^2}$$

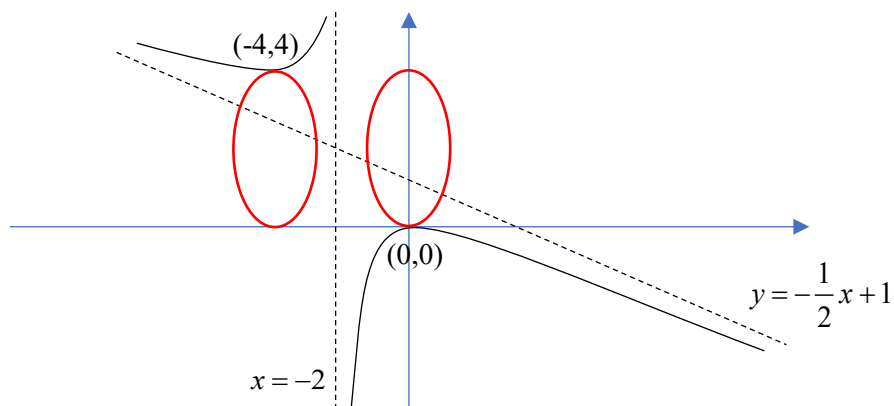
Since $(0,0)$ is a turning point, $0 = -\frac{1}{2} + \frac{4a+8}{16}$

$a = 0$

(b)

$$(x-d)^2 + \frac{1}{4}\left(\frac{-x^2}{2x+4} - 2\right)^2 = 1$$

$$(x-d)^2 + \left(\frac{y-2}{2}\right)^2 = 1$$

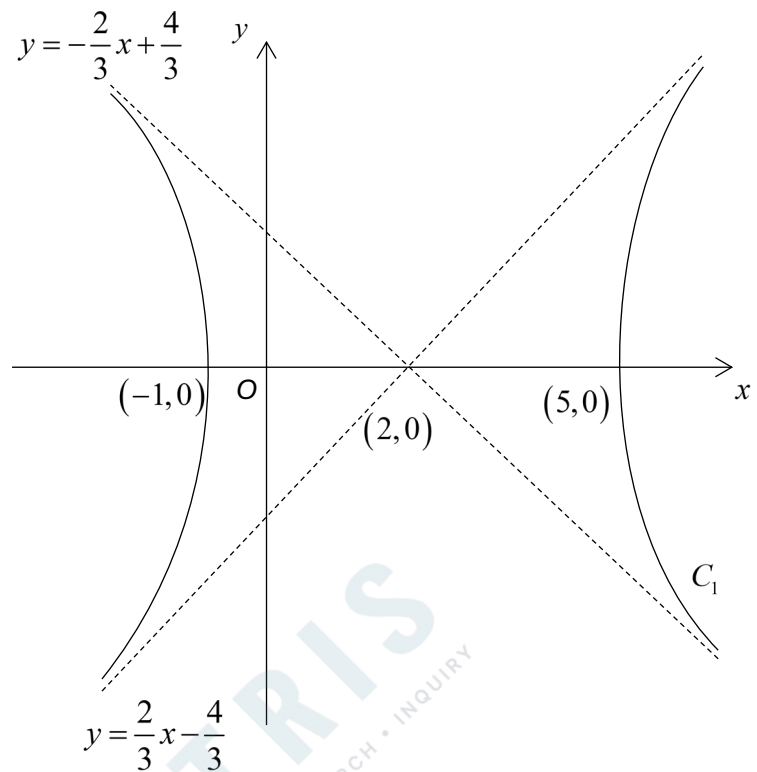
Since turning points are $(0,0)$ and $(-4,4)$ $d = -4$ or $d = 0$ (Note: Ellipse must meet C at exactly one point.)

5

(a) Finding asymptotes for C_1 :

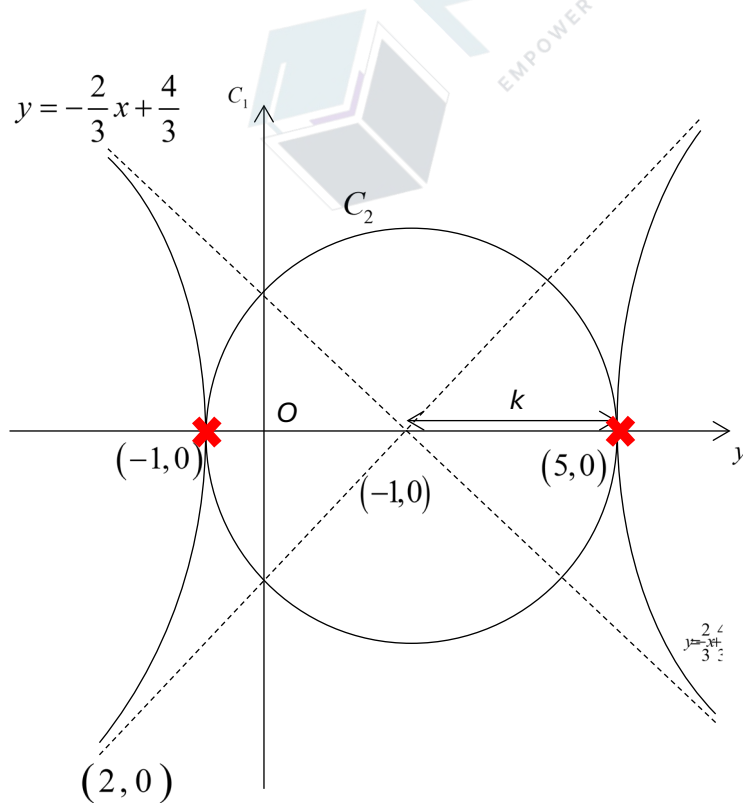
$$\frac{(x-2)^2}{3^2} - \frac{y^2}{2^2} = 0$$

$$y = \pm \frac{2}{3}(x-2)$$



(b) C_2 is a circle centred at $(2, 0)$ with radius k .

For C_1 and C_2 to intersect exactly twice, $k = 3$.



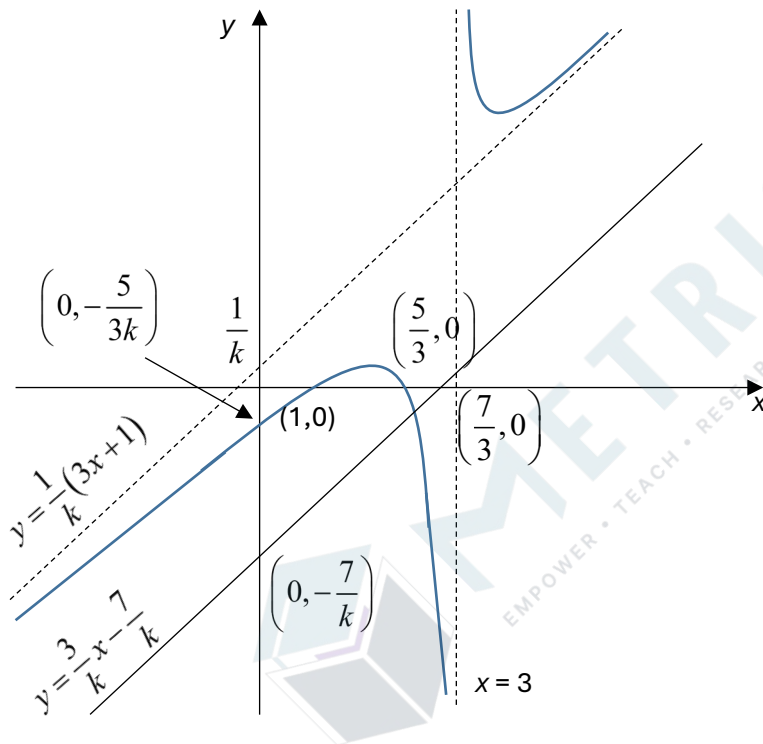
6

(a)(b)
$$y = \frac{-3x^2 + 8x - 5}{k(3-x)} = \frac{1}{k} \left(3x + 1 - \frac{8}{3-x} \right)$$

When $x = 0$, $y = -\frac{5}{3k}$

When $y = 0$, $-3x^2 + 8x - 5 = (-3x + 5)(x - 1) = 0 \Rightarrow x = 1$ or $\frac{5}{3}$

Equation of asymptotes: $y = \frac{1}{k}(3x + 1)$ and $x = 3$



(c)
$$\frac{1}{k} \left(3x + 1 - \frac{8}{3-x} \right) = \frac{3}{k}x - \frac{7}{k}$$

$$3x + 1 - \frac{8}{3-x} = 3x - 7$$

$$-\frac{8}{3-x} = -8$$

$$3 - x = 1 \Rightarrow x = 2$$

Therefore, from the graph, $x \leq 2$ or $x > 3$

7

(a) By observation, $d = 1$

$$\text{Hence, } y = x + 2 + \frac{n}{x-1} = \frac{(x+2)(x-1)+n}{x-1} = \frac{x^2 + x - 2 + n}{x-1}$$

By comparison, $a = b = 1$

(b) $y = \frac{x^2 + x + c}{x-1}$

$$\frac{dy}{dx} = \frac{(2x+1)(x-1) - (x^2 + x + c)}{(x-1)^2} = \frac{x^2 - 2x - 1 - c}{(x-1)^2}$$

For stationary points to occur, $\frac{dy}{dx} = 0$

$$x^2 - 2x - 1 - c = 0$$

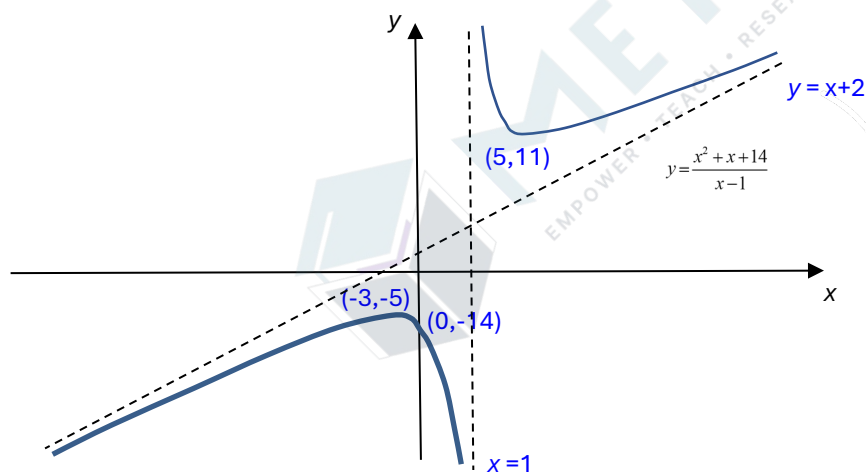
Hence, equation must yield 2 real roots, i.e $D > 0$

$$(-2)^2 - 4(1)(-c-1) > 0$$

$$4 + 4c + 4 > 0$$

$$c > -2$$

(c)



(d) Maximum number of solutions = 2

For 2 solutions to occur, $k > 8$

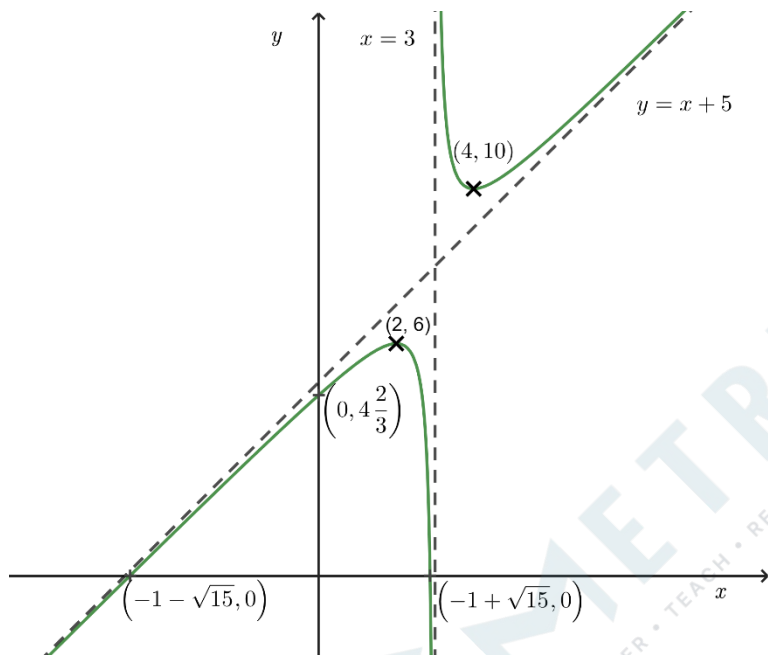
8

(i) Since $x = 3$, $c = -3$

$$y = x + 5 + \frac{d}{x-3} = \frac{x^2 + 2x - 15 + d}{x-3} = \frac{ax^2 + bx - 14}{x-3}$$

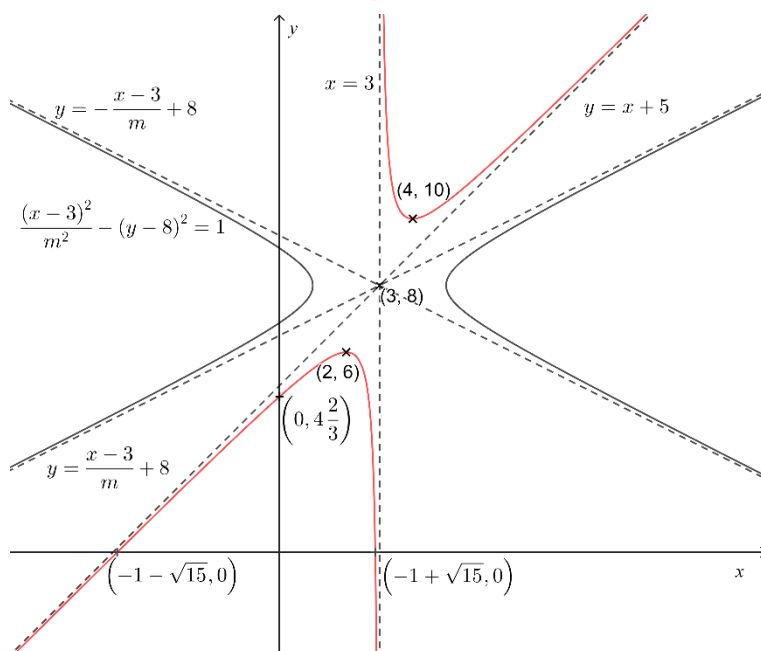
Therefore $a = 1$ and $b = 2$.

(ii) Graph of $y = \frac{x^2 + 2x - 14}{x-3}$



When rounded to 3 s.f, the axial intercepts are $(0, 4.67)$, $(-4.87, 0)$ and $(2.87, 0)$.

(iii)



(iv)

The asymptotes of $\frac{(x-3)^2}{m^2} - (y-8)^2 = 1$ ----(1) are

$$y = \frac{x-3}{m} + 8 \text{ and } y = -\frac{x-3}{m} + 8$$

$$y = \frac{x^2 + 2x - 14}{x-3} \text{ ----(2) has asymptotes } x = 3 \text{ and}$$

$$y = x + 5$$

All the asymptotes meet at the point (3,8).

Observing the graphs, (1) and (2) intersect when the asymptote of (1) with positive gradient,

$y = \frac{x-3}{m} + 8$ must be between the asymptotes of (2) i.e. $y = x + 5$ and $x = 3$.

Thus, gradient of $y = \frac{x-3}{m} + 8$ need to be between the gradient of $y = x + 5$ and gradient of the vertical line, $x = 3$.

$$1 < \frac{1}{m}$$

Therefore, $0 < m < 1$



9

(i) Consider $y = k$, k is a constant

$$\frac{x(x-a)}{x+a} = k$$

$$x^2 - ax = xk + ak$$

$$x^2 - (a+k)x - ak = 0$$

For the range of y can take, the line $y = k$ and the curve C should have point(s) of intersection.

$$(a+k)^2 + 4ak \geq 0$$

$$a^2 + 2ak + k^2 + 4ak \geq 0$$

$$a^2 + 6ak + k^2 \geq 0$$

Consider $k^2 + 6ak + a^2 = 0$

$$k = \frac{-6a \pm \sqrt{36a^2 - 4a^2}}{2} = \frac{-6a \pm \sqrt{32a^2}}{2} = (-3 \pm 2\sqrt{2})a$$

$$\therefore k \geq (-3 + 2\sqrt{2})a \text{ or } k \leq (-3 - 2\sqrt{2})a$$

Hence, $y \geq (-3 + 2\sqrt{2})a$ or $y \leq (-3 - 2\sqrt{2})a$

(ii) $y = \frac{x(x-1)}{x+1}$ and $y = -\frac{5}{2} + \frac{10}{x+3}$

$$\frac{x(x-1)}{x+1} = -\frac{5}{2} + \frac{10}{x+3}$$

$$2x(x-1)(x+3) = (-5(x+3) + 20)(x+1)$$

$$2x(x-1)(x+3) = (5-5x)(x+1)$$

$$2x(x-1)(x+3) = -5(x-1)(x+1)$$

$$(x-1)(2x(x+3) + 5(x+1)) = 0$$

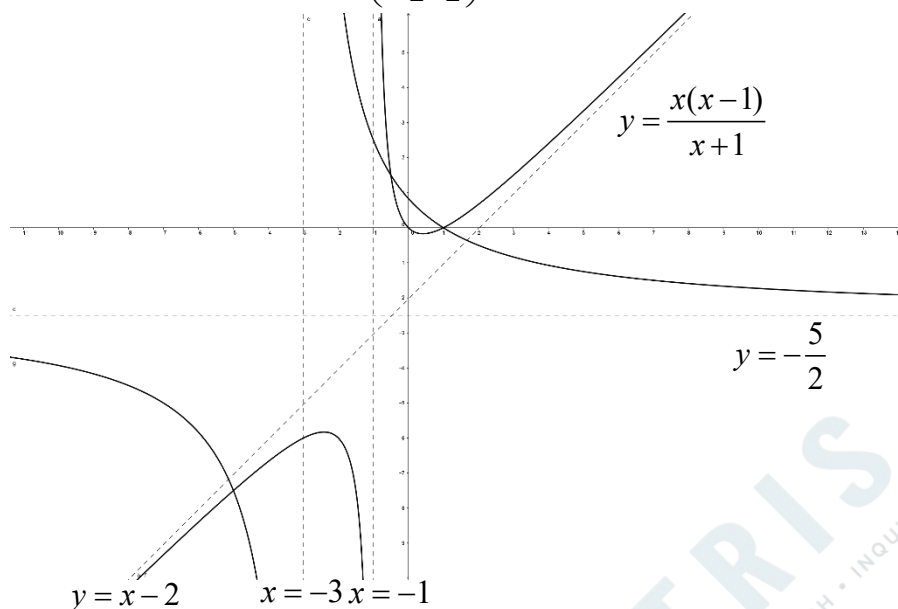
$$(x-1)(2x^2 + 11x + 5) = 0$$

$$(x-1)(2x+1)(x+5) = 0$$

$$x = 1 \text{ or } x = -\frac{1}{2} \text{ or } x = -5$$

(iii) $y = \frac{x(x-1)}{x+1}$ Sketch the curve $y = -\frac{5}{2} + \frac{10}{x+3}$

Coordinates of intersection $\left(-\frac{1}{2}, \frac{3}{2}\right)$ and $(1, 0)$ and $(-5, -15/2)$



Note: the minimum and maximum y values can be found from (i):
 $(0.414, 2\sqrt{2} - 3)$ and $(-2.41, -2\sqrt{2} - 3)$

(iii) Area

$$\begin{aligned}
 A &= \int_{-\frac{1}{2}}^1 \left(\left(-\frac{5}{2} + \frac{10}{x+3} \right) - \frac{x(x-1)}{x+1} \right) dx \\
 &= \int_{-\frac{1}{2}}^1 \left(-\frac{5}{2} + \frac{10}{x+3} - (x-2) - \frac{2}{x+1} \right) dx \\
 &= \int_{-\frac{1}{2}}^1 \left(-\frac{1}{2} - x - \frac{2}{x+1} + \frac{10}{x+3} \right) dx \\
 &= \left[-\frac{1}{2}x - \frac{x^2}{2} - 2\ln|x+1| + 10\ln|x+3| \right]_{-\frac{1}{2}}^1 \\
 &= -\frac{1}{2} - \frac{1^2}{2} - 2\ln|2| + 10\ln|4| - \left(\frac{1}{4} - \frac{\left(-\frac{1}{2}\right)^2}{2} - 2\ln\left|\frac{1}{2}\right| + 10\ln\left|\frac{5}{2}\right| \right) \\
 &= -\frac{9}{8} - 2\ln 2 + 20\ln 2 - 2\ln 2 - 10\ln 5 + 10\ln 2 \\
 &= 26\ln 2 - 10\ln 5 - \frac{9}{8} \\
 a &= 26, \quad b = -10, \quad c = -\frac{9}{8}
 \end{aligned}$$

10

(a)

$$C: y = \frac{ax^2 + x}{x+b}, \quad a \neq \frac{1}{b} \text{ and } a, b > 0.$$

$$\frac{dy}{dx} = \frac{(x+b)(2ax+1) - (ax^2+x)}{(x+b)^2} = \frac{ax^2 + 2abx + b}{(x+b)^2}$$

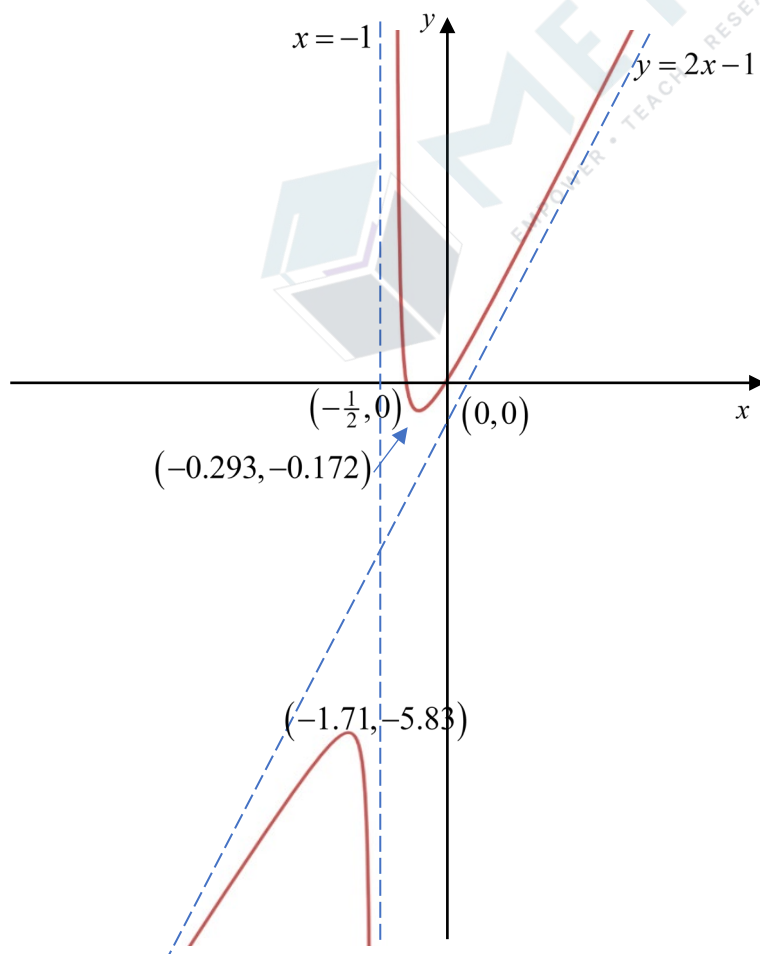
Since C has no stationary points, $\frac{dy}{dx} = 0$ has no real solutions. Hence, $ax^2 + 2abx + b = 0$ has no real solutions.

Thus the discriminant $(2ab)^2 - 4ab < 0$.

Since a and b are positive constants, on simplifying, the relationship is $0 < ab < 1$.

(b)

$$y = \frac{2x^2 + x}{x+1} = 2x - 1 + \frac{1}{x+1}$$



(c)

$$\text{Volume} = \pi \int_{-\frac{1}{2}}^0 \left(\frac{2x^2 + x}{x+1} \right)^2 dx = 0.0245 \text{ (3.s.f)}$$