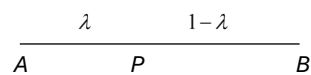
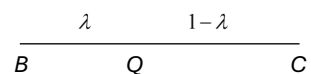


1

$$(a) \quad \overrightarrow{OP} = \frac{\lambda \overrightarrow{OB} + (1-\lambda) \overrightarrow{OA}}{\lambda + (1-\lambda)} = \lambda \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2-2\lambda \\ 1 \\ -1-\lambda \end{pmatrix}$$



$$\overrightarrow{OQ} = \frac{\lambda \overrightarrow{OC} + (1-\lambda) \overrightarrow{OB}}{\lambda + (1-\lambda)} = \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \lambda \\ 1+2\lambda \\ 2\lambda-2 \end{pmatrix}$$



$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 3\lambda-2 \\ 2\lambda \\ 3\lambda-1 \end{pmatrix}$$

$$(b) \quad \overrightarrow{PX} = \overrightarrow{OX} - \overrightarrow{OP} = \begin{pmatrix} 2 \\ -1 \\ -5/2 \end{pmatrix} - \begin{pmatrix} 2-2\lambda \\ 1 \\ -1-\lambda \end{pmatrix} = \begin{pmatrix} 2\lambda \\ -2 \\ \lambda-3/2 \end{pmatrix}$$

Since P, Q and X are collinear, $\overrightarrow{PX} = k \overrightarrow{PQ}$ for some $k \in \mathbb{R}, k \neq 0$

$$\begin{pmatrix} 2\lambda \\ -2 \\ \lambda-3/2 \end{pmatrix} = k \begin{pmatrix} 3\lambda-2 \\ 2\lambda \\ 3\lambda-1 \end{pmatrix}$$

$$2\lambda = k(3\lambda-2) \quad \text{--- (1)}$$

$$-2 = 2k\lambda \quad \text{--- (2)}$$

$$\lambda - \frac{3}{2} = k(3\lambda-1) \quad \text{--- (3)}$$

From (2), $k = -\frac{1}{\lambda}$

Sub $k = -\frac{1}{\lambda}$ into (1), $-2\lambda^2 = 3\lambda - 2$

$$2\lambda^2 + 3\lambda - 2 = 0$$

$$\lambda = \frac{1}{2} \text{ or } \lambda = -2 \text{ (NA } \because 0 < \lambda < 1)$$

Checking,

Sub $k = -\frac{1}{\lambda}$ into (3), $-\lambda^2 + \frac{3}{2}\lambda = 3\lambda - 1$

$$2\lambda^2 + 3\lambda - 2 = 0$$

$$\lambda = \frac{1}{2} \text{ or } \lambda = -2 \text{ (NA } \because 0 < \lambda < 1)$$

$$\therefore \lambda = \frac{1}{2}$$

2

(a) $\underline{r} - \underline{q}$ is parallel to $\underline{p}$

$$\Rightarrow \underline{r} - \underline{q} = \lambda \underline{p}$$

$$\underline{r} = \underline{q} + \lambda \underline{p}$$

The point R lies on a line that passes through the point Q and is parallel to the vector $\underline{p}$.

(b) $(\underline{r} - \underline{q}) \cdot \underline{p} = 0$

$$\underline{r} \cdot \underline{p} - \underline{q} \cdot \underline{p} = 0$$

$$\underline{r} \cdot \underline{p} = \underline{q} \cdot \underline{p}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$$

$$2x - 5y + 3z = -7$$

The point R lies on a plane that contains the point Q and is perpendicular to the vector $\underline{p}$.



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3

- (a)(i) $\underline{v} \times \underline{u} = \underline{0}$
- Either $\underline{v} = \underline{0}$
 - Or $\underline{v} \neq \underline{0}$ (and given $\underline{u} \neq \underline{0}$), $\underline{v}$ and $\underline{u}$ are parallel
- Combining both cases, $\underline{v} = k\underline{u}$, where $k \in \mathbb{R}$

(a)(ii) $\underline{v} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix},$

$$\underline{n} = \frac{1}{\sqrt{1+4+4}} \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

Alternative Solution: $\underline{n} = -\frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

(b)(i) $\overrightarrow{AB} = \underline{b} - \underline{a}$

$$\overrightarrow{AC} = \left(4\underline{a} - \frac{2}{3}\underline{b} \right) - \underline{a} = 3\underline{a} - \frac{2}{3}\underline{b}$$

$$\text{Area of } \triangle ABC = 14$$

$$\frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = 14$$

$$\frac{1}{2} \left| (\underline{b} - \underline{a}) \times \left(3\underline{a} - \frac{2}{3}\underline{b} \right) \right| = 14$$

$$\left| 3\underline{b} \times \underline{a} - \frac{2}{3}\underline{b} \times \underline{b} - 3\underline{a} \times \underline{a} + \frac{2}{3}\underline{a} \times \underline{b} \right| = 28$$

Since $\underline{a} \times \underline{a} = \underline{0}$, $\underline{b} \times \underline{b} = \underline{0}$ and $\underline{b} \times \underline{a} = -\underline{a} \times \underline{b}$,

$$\left| -\frac{7}{3}\underline{a} \times \underline{b} \right| = 28$$

$$\frac{7}{3} |\underline{a} \times \underline{b}| = 28$$

$$|\underline{a} \times \underline{b}| = 12 \text{ (shown)}$$

(b)(ii) Since $|\underline{a}| = 5$, $|\underline{b}| = 3$

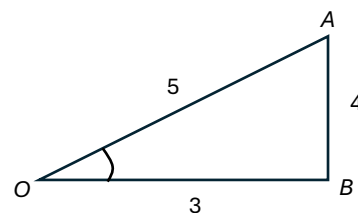
$$|\underline{a} \times \underline{b}| = 12 \Rightarrow |\underline{a}| |\underline{b}| \sin \angle AOB = 12$$

$$15 \sin \angle AOB = 12$$

$$\sin \angle AOB = \frac{4}{5}$$

$\angle AOB$ is obtuse $\Rightarrow \cos \angle AOB < 0$

$$\cos \angle AOB = -\frac{3}{5}$$



4 DHS Prelim 9758/2023/01/Q5

- (a) Since A, B and C are collinear, $BC = kAB$ where $k \in \mathbb{R}$

$$\overrightarrow{BC} = k(\underline{b} - \underline{a})$$

$$\text{Given } \overrightarrow{BC} = 3\underline{b} - \mu\underline{a},$$

$$3\underline{b} - \mu\underline{a} = k(\underline{b} - \underline{a})$$

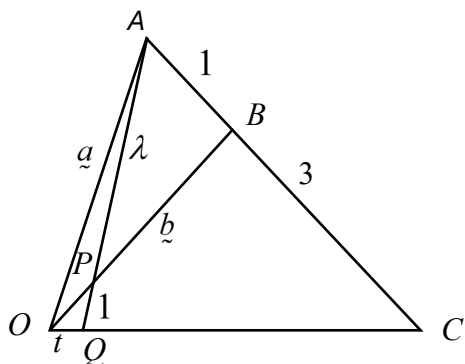
$$\text{By observation, } \mu = k = 3$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$$3\underline{b} - 3\underline{a} = \overrightarrow{OC} - \underline{b} \quad \text{hell}$$

$$\overrightarrow{OC} = 4\underline{b} - 3\underline{a}$$

- (b)



Using ratio theorem

$$\overrightarrow{OP} = \frac{\lambda \overrightarrow{OQ} + \overrightarrow{OA}}{1 + \lambda}$$

$$= \frac{1}{1 + \lambda} [\lambda t \overrightarrow{OC} + \underline{a}]$$

$$= \frac{1}{1 + \lambda} [\lambda t (4\underline{b} - 3\underline{a}) + \underline{a}]$$

$$= \frac{1}{1 + \lambda} [4\lambda t \underline{b} + (1 - 3\lambda t) \underline{a}]$$

$$\text{Since } \overrightarrow{OP} \parallel \underline{b}, \quad 1 - 3\lambda t = 0$$

$$\lambda t = \frac{1}{3}$$

- (c) When $\lambda = 5$,

$$\overrightarrow{OP} = \frac{1}{1 + \lambda} (4\lambda t) \underline{b} = \frac{1}{1 + 5} \left(\frac{4}{3} \right) \underline{b} = \frac{2}{9} \underline{b}$$

$$\therefore OP : PB = \frac{2}{9} : \frac{7}{9} = 2 : 7$$

5

- (a) By ratio theorem, $\overrightarrow{OM} = \frac{2a+c}{3}$.

Let a normal to Π be $\overrightarrow{AC} = \underline{c} - \underline{a}$.

$$\underline{r} \cdot (\underline{c} - \underline{a}) = \left(\frac{2\underline{a} + \underline{c}}{3} \right) \cdot (\underline{c} - \underline{a})$$

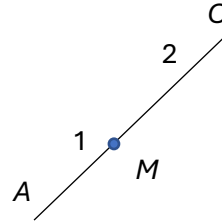
$$3r_{\sim} \cdot (\mathbf{c} - \mathbf{a}) = (2\mathbf{a} + \mathbf{c}) \cdot (\mathbf{c} - \mathbf{a})$$

$$\underline{r} \cdot (3\underline{c} - 3\underline{a}) = (2\underline{a} + \underline{c}) \cdot (\underline{c} - \underline{a})$$

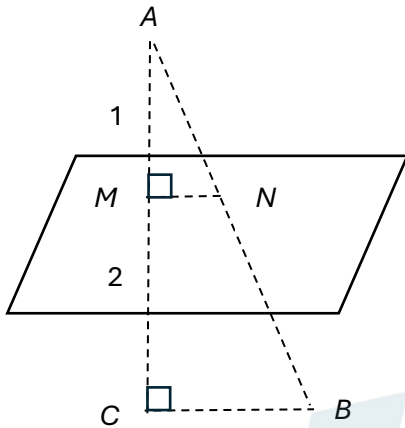
$$= 2\underset{\sim}{a} \cdot \underset{\sim}{c} - 2\underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{c} \cdot \underset{\sim}{c} - \text{c.c.}$$

$$= 2\underset{\sim}{a} \cdot \underset{\sim}{c} - 2\underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{c}$$

$$= \underline{a} \cdot \underline{c} - 2\underline{a} \cdot \underline{a} + \underline{c} \cdot \underline{c} \quad (\text{shown})$$



- (b)**



Since $\angle ACB = 90^\circ$ and AC is perpendicular to Π ,
 $\angle AMN = 90^\circ$

$$\frac{\text{Area of triangle } AMN}{\text{Area of triangle } ABC} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$$\therefore \frac{\text{Area of triangle } AMN}{\text{Area of quadrilateral } MNBC} = \frac{1}{8}$$

6

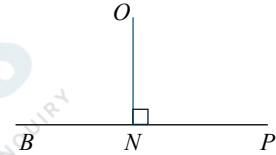
$$\begin{aligned}
 \text{(a)} \quad BP &= OP - OB & AP &= OP - OA \\
 &= (3\vec{a} - 2\vec{b}) - \vec{b} & &= (3\vec{a} - 2\vec{b}) - \vec{a} \\
 &= 3\vec{a} - 3\vec{b} & \text{and} &= 2\vec{a} - 2\vec{b} \\
 &= 3(\vec{a} - \vec{b}) & &= 2(\vec{a} - \vec{b})
 \end{aligned}$$

Since $\overrightarrow{BP} = \frac{3}{2}\overrightarrow{AP}$, points A , B and P are collinear (shown)

$$BP : AP = 3 : 2$$

$$\text{(b)} \quad l_{BP} : \vec{r} = \vec{b} + \lambda(\vec{a} - \vec{b}), \lambda \in \mathbb{R}$$

Since N lies on line BP , $\overrightarrow{ON} = \vec{b} + \lambda(\vec{a} - \vec{b})$ for some $\lambda \in \mathbb{R}$.



Since $\overrightarrow{ON}$ is perpendicular to l_{BP} , $\overrightarrow{ON}$ is perpendicular to $(\vec{a} - \vec{b})$.

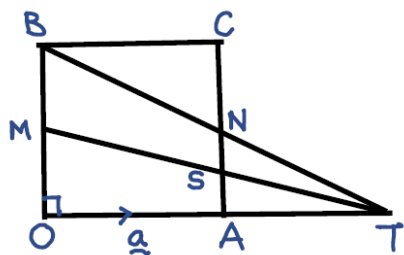
$$\begin{aligned}
 &[\vec{b} + \lambda(\vec{a} - \vec{b})] \cdot (\vec{a} - \vec{b}) = 0 \\
 &[(1 - \lambda)\vec{b} + \lambda\vec{a}] \cdot (\vec{a} - \vec{b}) = 0 \\
 (1 - \lambda)(\vec{b} \cdot \vec{a}) - (1 - \lambda)(\vec{b} \cdot \vec{b}) + \lambda(\vec{a} \cdot \vec{a}) - \lambda(\vec{a} \cdot \vec{b}) &= 0 \\
 (1 - \lambda)(\vec{b} \cdot \vec{a}) - (1 - \lambda)|\vec{b}|^2 + \lambda|\vec{a}|^2 - \lambda(\vec{a} \cdot \vec{b}) &= 0
 \end{aligned}$$

Given that $|\vec{b}| = 1$, $|\vec{a}| = \frac{4}{3}$, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\frac{\pi}{3} = \left(\frac{4}{3}\right)(1)\cos\frac{\pi}{3} = \frac{2}{3}$,

$$\begin{aligned}
 (1 - \lambda)\left(\frac{2}{3}\right) - (1 - \lambda)(1)^2 + \lambda\left(\frac{4}{3}\right) - \lambda\left(\frac{2}{3}\right) &= 0 \\
 (1 - \lambda)\left(-\frac{1}{3}\right) + \frac{16}{9}\lambda - \frac{2}{3}\lambda &= 0 \\
 -\frac{1}{3} + \frac{1}{3}\lambda + \frac{10}{9}\lambda &= 0 \\
 \frac{13}{9}\lambda &= \frac{1}{3} \\
 \lambda &= \frac{3}{13}
 \end{aligned}$$

$$\begin{aligned}
 \overrightarrow{BN} &= \overrightarrow{ON} - \overrightarrow{OB} \\
 &= \left[\vec{b} + \frac{3}{13}(\vec{a} - \vec{b})\right] - \vec{b} \\
 &= \frac{3}{13}(\vec{a} - \vec{b}) \\
 &= \frac{3}{13}\overrightarrow{BA} \\
 \therefore \frac{BN}{BA} &= \frac{3}{13}.
 \end{aligned}$$

(i)



By ratio theorem,

$$\overrightarrow{OS} = \frac{\overrightarrow{OC} + 3\overrightarrow{OA}}{1+3}$$

$$\overrightarrow{OS} = \frac{1}{4}\overrightarrow{OC} + \frac{3}{4}\overrightarrow{OA}$$

$$\begin{aligned}\overrightarrow{OS} &= \frac{1}{4}(\underline{a} + \underline{b}) + \frac{3}{4}\underline{a} && \text{Since } \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} \\ &= \overrightarrow{OA} + \overrightarrow{OB} \\ &= \underline{a} + \underline{b}\end{aligned}$$

$$\overrightarrow{OS} = \underline{a} + \frac{1}{4}\underline{b}$$

$$(ii) \quad \overrightarrow{MS} = \overrightarrow{OS} - \overrightarrow{OM}$$

$$= \left(\underline{a} + \frac{1}{4}\underline{b} \right) - \frac{1}{2}\underline{b}$$

$$= \underline{a} - \frac{1}{4}\underline{b}$$

$$l_{MS} : \underline{r} = \overrightarrow{OM} + \lambda \overrightarrow{MS}$$

$$\underline{r} = \frac{1}{2}\underline{b} + \lambda \left(\underline{a} - \frac{1}{4}\underline{b} \right), \lambda \in \mathbb{R}$$

$$\overrightarrow{BN} = \overrightarrow{ON} - \overrightarrow{OB}$$

$$= \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC})$$

$$= \frac{1}{2}[\underline{a} + (\underline{a} + \underline{b})] - \underline{b}$$

$$= \underline{a} - \frac{1}{2}\underline{b}$$

$$l_{BN} : \underline{r} = \overrightarrow{OB} + \mu \overrightarrow{BN}$$

$$\underline{r} = \underline{b} + \mu \left(\underline{a} - \frac{1}{2}\underline{b} \right), \mu \in \mathbb{R}$$

(iii) At T , $\frac{1}{2}\vec{b} + \lambda\left(\vec{a} - \frac{1}{4}\vec{b}\right) = \vec{b} + \mu\left(\vec{a} - \frac{1}{2}\vec{b}\right)$

Since $\vec{a}$ and $\vec{b}$ are non parallel and non zero,
comparing coefficients of $\vec{a}$,

$$\lambda = \mu$$

comparing coefficients of $\vec{b}$,

$$\frac{1}{2} - \frac{\lambda}{4} = 1 - \frac{\mu}{2}$$

Solving, $\frac{1}{2} - \frac{\lambda}{4} = 1 - \frac{\lambda}{2}$

$$\frac{\lambda}{4} = \frac{1}{2}$$

$$\lambda = 2$$

$$\mu = \lambda = 2$$

$$\begin{aligned}\therefore \vec{OT} &= \frac{1}{2}\vec{b} + 2\left(\vec{a} - \frac{1}{4}\vec{b}\right) \\ &= 2\vec{a}\end{aligned}$$

Since $\vec{OT} = 2\vec{OA}$,
then O , T and A are collinear points with O as the common point.



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(i) $|\underline{p} \cdot \underline{q}|$ is the length of projection of $\underline{q}$ onto $\underline{p}$.

(ii) Since M lies on line OP , $\overrightarrow{OM} = \lambda \underline{p}$ for some λ .

Since M is the foot of the perpendicular from Q to the line OP ,

$$\overrightarrow{QM} \cdot \overrightarrow{OP} = 0$$

$$(\overrightarrow{OM} - \overrightarrow{OQ}) \cdot \overrightarrow{OP} = 0$$

$$(\lambda \underline{p} - \underline{q}) \cdot \underline{p} = 0$$

$$\lambda \underline{p} \cdot \underline{p} = \underline{q} \cdot \underline{p}$$

$$\lambda |\underline{p}|^2 = \underline{q} \cdot \underline{p}$$

$$\lambda = 2 \text{ (as } \underline{p} \text{ is a unit vector)}$$

$$\therefore \overrightarrow{OM} = 2\underline{p} \text{ (shown)}$$

(iii) $\overrightarrow{OM} = 2\underline{p}$; $\overrightarrow{OQ} = \underline{q}$; $\overrightarrow{OR} = \underline{r} = k(\underline{p} - \underline{q})$

$$\overrightarrow{QM} = 2\underline{p} - \underline{q}$$

$$\overrightarrow{QR} = k(\underline{p} - \underline{q}) - \underline{q} = k\underline{p} - (1+k)\underline{q}$$

Area of triangle RQM

$$= \frac{1}{2} |\overrightarrow{QM} \times \overrightarrow{QR}|$$

$$= \frac{1}{2} |(2\underline{p} - \underline{q}) \times (k\underline{p} - (1+k)\underline{q})|$$

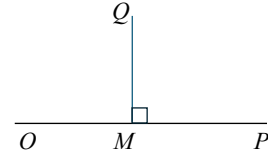
$$= \frac{1}{2} |2k(\underline{p} \times \underline{p}) - 2(1+k)(\underline{p} \times \underline{q}) - k(\underline{q} \times \underline{p}) + (1+k)(\underline{q} \times \underline{q})| \quad (\text{since } \underline{p} \times \underline{p} = \underline{q} \times \underline{q} = \underline{0})$$

$$= \frac{1}{2} |-2(1+k)(\underline{p} \times \underline{q}) + k(\underline{p} \times \underline{q})| \quad (\text{since } -\underline{q} \times \underline{p} = \underline{p} \times \underline{q})$$

$$= \frac{1}{2} |(-2-k)(\underline{p} \times \underline{q})|$$

$$= \frac{|2+k|}{2} |\underline{p} \times \underline{q}| \text{ where } a = \frac{|2+k|}{2}$$

When $k = -2$, points R , Q and M are collinear.



9

- (a) It describes the line passing through A and parallel to l .

OR

It describes the line passing through A and parallel to $\underline{d}$.

- (b) Let the foot of perpendicular be N . Then $\overrightarrow{ON} = s\underline{d}$ for some s .

Since $\overrightarrow{AN}$ is perpendicular to l ,

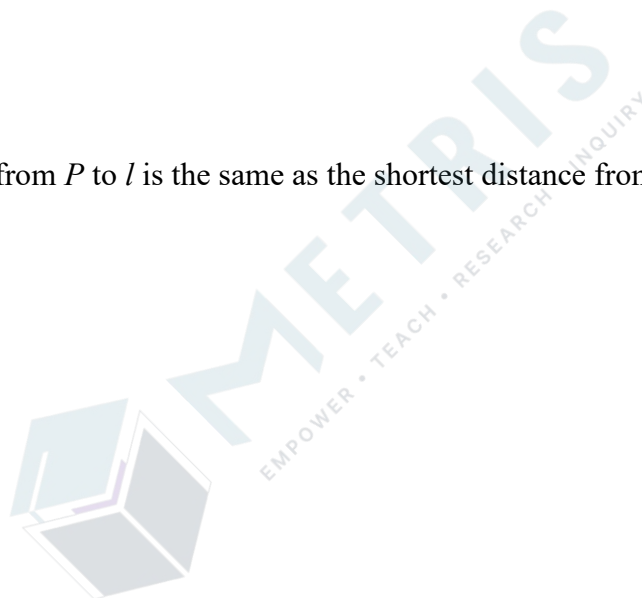
$$(s\underline{d} - \underline{a}) \cdot \underline{d} = 0$$

$$s(\underline{d} \cdot \underline{d}) = \underline{a} \cdot \underline{d}$$

$$s = \frac{\underline{a} \cdot \underline{d}}{|\underline{d}|^2} \quad (\because |\underline{d}| = 1)$$

$$\therefore \overrightarrow{ON} = s\underline{d} = (\underline{a} \cdot \underline{d})\underline{d}$$

- (c) The shortest distance from P to l is the same as the shortest distance from A to l . (or consider area of parallelogram)



10

Equation of line: $\underline{r} = \underline{a} + \lambda \underline{n}$

$$(\underline{a} + \lambda \underline{n}) \cdot \underline{n} = d$$

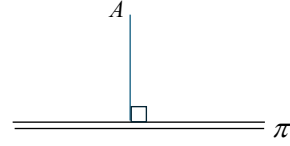
$$\underline{a} \cdot \underline{n} + \lambda \underline{n} \cdot \underline{n} = d$$

$$\lambda |\underline{n}|^2 = d - \underline{a} \cdot \underline{n}$$

$$\lambda = \frac{d - |\underline{a}| |\underline{n}| \cos 30^\circ}{|\underline{n}|^2}$$

$$= \frac{d - \frac{4\sqrt{3}}{2}}{2}$$

$$= d - 2\sqrt{3}$$



Position vector of the foot of perpendicular F is $\overrightarrow{OF} = \underline{a} + (d - 2\sqrt{3})\underline{n}$

$$\overrightarrow{AF} = \overrightarrow{OF} - \overrightarrow{OA} = (d - 2\sqrt{3})\underline{n}$$

Perpendicular distance of point A to plane

$$= |\overrightarrow{AF}|$$

$$= |d - 2\sqrt{3}|$$

Method ①:

Area of triangle OAF

$$= \frac{1}{2} |\underline{OA}| |\underline{AF}| \sin 30^\circ$$

$$= \frac{1}{2} (4) |d - 2\sqrt{3}| \left(\frac{1}{2} \right)$$

$$= |d - 2\sqrt{3}|$$

Method ②:

Area of triangle OAF

$$= \frac{1}{2} |\overrightarrow{OA} \times \overrightarrow{OF}|$$

$$= \frac{1}{2} \left| \underline{a} \times [\underline{a} + (d - 2\sqrt{3})\underline{n}] \right|$$

$$= \frac{1}{2} \left| (\underline{a} \times \underline{a}) + (d - 2\sqrt{3})(\underline{a} \times \underline{n}) \right|$$

$$= \frac{1}{2} |d - 2\sqrt{3}| |\underline{a}| |\underline{n}| \sin 30^\circ$$

$$= \frac{1}{2} |d - 2\sqrt{3}| 4(1) \left(\frac{1}{2} \right)$$

$$= |d - 2\sqrt{3}|$$

11

- (i) Since Q lies on the line passing through OB , OQ is parallel to OB . Hence Q has position vector in the form $\lambda \underline{b}$ where λ is a real constant.

Alternative Solution:

Equation of line OB : $\underline{r} = \underline{0} + \lambda \underline{b}, \lambda \in \mathbb{R}$

$$\underline{r} = \lambda \underline{b}$$

Since Q lies on the line, it has position vector in the form $\lambda \underline{b}$, where λ is a real constant.

(ii)



By ratio theorem, $\overrightarrow{OP} = \frac{1}{4}(3\underline{a} + \underline{b})$

$$\overrightarrow{OQ} = \lambda \underline{b}$$

$$\overrightarrow{AQ} = \lambda \underline{b} - \underline{a}$$

Given $AQ \perp OP$,

$$\overrightarrow{AQ} \cdot \overrightarrow{OP} = 0$$

$$\frac{1}{4}(3\underline{a} + \underline{b}) \cdot (\lambda \underline{b} - \underline{a}) = 0$$

$$3\lambda \underline{a} \cdot \underline{b} - 3\underline{a} \cdot \underline{a} + \lambda \underline{b} \cdot \underline{b} - \underline{a} \cdot \underline{b} = 0$$

$$3\lambda |\underline{a}| |\underline{b}| \cos \theta - 3|\underline{a}|^2 + \lambda |\underline{b}|^2 - |\underline{a}| |\underline{b}| \cos \theta = 0$$

$$3\lambda \cos \theta - 3 + \lambda - \cos \theta = 0$$

$$(3\cos \theta + 1)\lambda = 3 + \cos \theta$$

$$\lambda = \frac{3 + \cos \theta}{3\cos \theta + 1}$$

(iii) **Analytical Method:**

$$\lambda = \frac{3 + \cos \theta}{3 \cos \theta + 1} = \frac{1}{3} + \frac{\frac{8}{3}}{3 \cos \theta + 1}$$

$$\begin{array}{r} \frac{1}{3} \\ 3 \cos \theta + 1 \overline{) \cos \theta + 3} \\ \underline{-(\cos \theta + \frac{1}{3})} \\ \frac{8}{3} \end{array}$$

$$0 < \theta < \frac{\pi}{2}$$

$$0 < \cos \theta < 1$$

$$0 < 3 \cos \theta < 3$$

$$1 < 3 \cos \theta + 1 < 4$$

$$\frac{1}{4} < \frac{1}{3 \cos \theta + 1} < 1$$

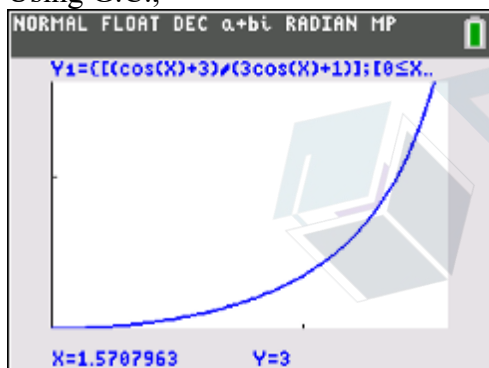
$$\frac{2}{3} < \frac{\frac{8}{3}}{3 \cos \theta + 1} < \frac{8}{3}$$

$$1 < \frac{1}{3} + \frac{\frac{8}{3}}{3 \cos \theta + 1} < 3$$

$$1 < \lambda < 3$$

Graphical Method:

Using G.C.,



$$\therefore 1 < \lambda < 3$$

Q lies on OB produced.

Hence, the point Q does not lie between O and B .

12

$$(a)(i) \quad (3\hat{b} - 2\hat{a}) \cdot (3\hat{b} - 2\hat{a}) = 9(\hat{b} \cdot \hat{b}) - 6(\hat{b} \cdot \hat{a}) - 6(\hat{a} \cdot \hat{b}) + 4(\hat{a} \cdot \hat{a})$$

$$|3\hat{b} - 2\hat{a}|^2 = 9|\hat{b}|^2 - 12(\hat{a} \cdot \hat{b}) + 4|\hat{a}|^2$$

$$|3\hat{b} - 2\hat{a}|^2 = 9(2)^2 - 12(\hat{a} \cdot \hat{b}) + 4(1)^2$$

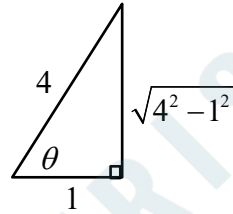
$$\hat{a} \cdot \hat{b} = \frac{40 - 34}{12}$$

$$= \frac{1}{2}$$

(a)(ii) Now, $\hat{a} \cdot \hat{b} = |\hat{a}||\hat{b}|\cos\theta$ where θ is the angle between $\hat{a}$ and $\hat{b}$.

$$\text{Hence, } \cos\theta = \frac{\hat{a} \cdot \hat{b}}{|\hat{a}||\hat{b}|} = \frac{0.5}{(1)(2)} = \frac{1}{4}.$$

$$\sin\theta = \frac{\sqrt{4^2 - 1^2}}{4} = \frac{\sqrt{15}}{4}$$



$$\therefore |\hat{a} \times \hat{b}| = |\hat{a}||\hat{b}|\sin\theta = (1)(2)\frac{\sqrt{15}}{4} = \frac{\sqrt{15}}{2}$$

$|\hat{a} \times \hat{b}|$ is the area of parallelogram with adjacent sides OA and OB

OR

$|\hat{a} \times \hat{b}|$ is twice the area of triangle OAB

(b) By ratio theorem,

$$\overrightarrow{OP} = \frac{2\vec{b} + \vec{a}}{3} \dots (1)$$

$$\overrightarrow{OM} = \frac{\vec{b} + \vec{c}}{2} \dots (2)$$

$OBMN$ forms a rhombus

$$\begin{aligned} \overrightarrow{ON} &= \overrightarrow{BM} \\ &= \overrightarrow{OM} - \vec{b} \\ &= \frac{\vec{b} + \vec{c}}{2} - \vec{b} \\ &= \frac{\vec{c} - \vec{b}}{2} \dots (3) \end{aligned}$$

P is a mid-point of MN

$$\overrightarrow{OP} = \frac{1}{2}(\overrightarrow{OM} + \overrightarrow{ON}) \dots (4)$$

Method ①: Equating two different expressions for $\overrightarrow{OP}$

Put (1), (2), (3) into (4):

$$\frac{2\vec{b} + \vec{a}}{3} = \frac{1}{2} \left[\left(\frac{\vec{b} + \vec{c}}{2} \right) + \left(\frac{\vec{c} - \vec{b}}{2} \right) \right]$$

$$\frac{2\vec{b} + \vec{a}}{3} = \frac{1}{2} \vec{c}$$

$$4\vec{b} + 2\vec{a} = 3\vec{c}$$

$$2\vec{a} + 4\vec{b} - 3\vec{c} = 0 \text{ (shown)}$$

Method ②: Equating two different expressions for $\overrightarrow{ON}$

From (4), $\overrightarrow{ON} = 2\overrightarrow{OP} - \overrightarrow{OM}$

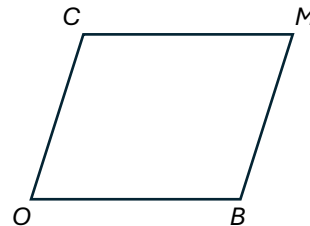
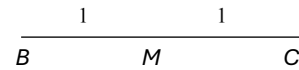
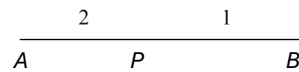
$$\left(\frac{\vec{c} - \vec{b}}{2} \right) = 2 \left(\frac{2\vec{b} + \vec{a}}{3} \right) - \left(\frac{\vec{b} + \vec{c}}{2} \right) \text{ [from (1), (2) and (3)]}$$

$$\frac{\vec{c} - \vec{b}}{2} = \frac{2\vec{a}}{3} + \frac{5\vec{b}}{6} - \frac{\vec{c}}{2}$$

$$\frac{2\vec{a}}{3} + \frac{5\vec{b}}{6} + \frac{\vec{b}}{2} - \frac{\vec{c}}{2} - \frac{\vec{c}}{2} = 0$$

$$\frac{2\vec{a}}{3} + \frac{4\vec{b}}{3} - \vec{c} = 0$$

$$2\vec{a} + 4\vec{b} - 3\vec{c} = 0 \text{ (shown)}$$



13

(a)(i) Given $\underline{r} \cdot \underline{q} = \underline{p} \cdot \underline{q}$.

R represents any point on the plane that is perpendicular to $\underline{q}$ and containing the point P .

(a)(ii) Given $\underline{r} \times \underline{q} = \underline{p} \times \underline{q}$

$$(\underline{r} \times \underline{q}) - (\underline{p} \times \underline{q}) = \underline{0}$$

$$(\underline{r} - \underline{p}) \times \underline{q} = \underline{0}$$

$$\underline{r} - \underline{p} // \underline{q}$$

$$\underline{r} - \underline{p} = k\underline{q}, k \in \mathbb{R}$$

$$\therefore \underline{r} = \underline{p} + k\underline{q}$$

R represents any point on the line containing the point P and parallel to $\underline{q}$.

Given that $AC : CB = 3 : 2$. By ratio theorem, $b = \frac{c+2a}{3} \Rightarrow c = 3b - 2a$.

Consider $|2\underline{a} - \underline{c}|^2 = (2\underline{a} - \underline{c}) \cdot (2\underline{a} - \underline{c})$

$$|2\underline{a} - \underline{c}|^2 = 4\underline{a} \cdot \underline{a} - 2\underline{a} \cdot \underline{c} - \underline{c} \cdot 2\underline{a} + \underline{c} \cdot \underline{c}$$

$$|2\underline{a} - \underline{c}|^2 = 4|\underline{a}|^2 - 4\underline{a} \cdot \underline{c} + |\underline{c}|^2 \quad (\text{Since } \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{a}, \underline{a} \cdot \underline{a} = |\underline{a}|^2, \underline{c} \cdot \underline{c} = |\underline{c}|^2)$$

$$|2\underline{a} - \underline{c}|^2 = 4|\underline{a}|^2 - 4\underline{a} \cdot (3\underline{b} - 2\underline{a}) + |\underline{c}|^2$$

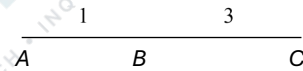
$$|2\underline{a} - \underline{c}|^2 = 4|\underline{a}|^2 - 12\underline{a} \cdot \underline{b} + 8|\underline{a}|^2 + |\underline{c}|^2$$

$$|2\underline{a} - \underline{c}|^2 = 12|\underline{a}|^2 - 12|\underline{a}||\underline{b}|\cos 60^\circ + |\underline{c}|^2$$

$$|2\underline{a} - \underline{c}|^2 = 12(1)^2 - 12(1)(1)\left(\frac{1}{2}\right) + (\sqrt{7})^2$$

$$|2\underline{a} - \underline{c}|^2 = 13$$

$$\therefore |2\underline{a} - \underline{c}| = \sqrt{13} \quad \text{since } |2\underline{a} - \underline{c}| \geq 0$$



14

(a) $\vec{p} = \lambda \vec{a} + (1 - \lambda) \vec{b}$

$$\vec{p} = \lambda \vec{a} + \vec{b} - \lambda \vec{b}$$

$$\vec{p} - \vec{b} = \lambda (\vec{a} - \vec{b})$$

$$\vec{BP} = \lambda \vec{BA} \text{ and common point B}$$

$\therefore$ the points A , B and P are collinear (shown).

(b) Let F be the foot of perpendicular from O to AB .

$$\vec{OF} = \vec{a} + \mu (\vec{b} - \vec{a}) \text{ for some } \mu$$

$$\vec{OF} \cdot \vec{AB} = 0$$

$$[\vec{a} + \mu (\vec{b} - \vec{a})] \cdot (\vec{b} - \vec{a}) = 0$$

$$\vec{a} \cdot (\vec{b} - \vec{a}) + \mu (\vec{b} - \vec{a}) \cdot (\vec{b} - \vec{a}) = 0$$

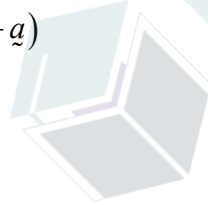
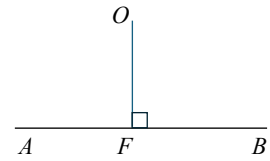
$$\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{a} + \mu |\vec{b} - \vec{a}|^2 = 0$$

$$|\vec{a}| |\vec{b}| \cos 90^\circ - |\vec{a}|^2 + \mu |\vec{b} - \vec{a}|^2 = 0$$

$$\mu |\vec{b} - \vec{a}|^2 = |\vec{a}|^2$$

$$\mu = \frac{|\vec{a}|^2}{|\vec{b} - \vec{a}|^2}$$

$$\therefore \vec{OF} = \vec{a} + \frac{|\vec{a}|^2}{|\vec{b} - \vec{a}|^2} (\vec{b} - \vec{a})$$



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15

(a)

$$\vec{OA} \cdot \vec{OB} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} = 0$$

$\therefore OA$ and OB are perpendicular.

$$\therefore \angle AOB = 90^\circ$$

(b)

$$\sin \theta = \frac{1}{2}$$

$$\therefore \theta = 30^\circ \text{ or } \theta = 150^\circ$$

Method ①: (using right-angle $\triangle OAB$)

Since $OA \perp OB$, $\therefore \triangle OAB$ is a right-angle $\triangle$.

Hence $\angle OBA$ and $\angle OAB$ must be acute.

$$\therefore \theta = 30^\circ$$

Method ②: (using dot product $\vec{AB} \cdot \vec{OB}$)

$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ \alpha \end{pmatrix}$$

$$\text{Since } \vec{AB} \cdot \vec{OB} = \begin{pmatrix} -2 \\ 1 \\ \alpha \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} = \alpha^2 > 0, \text{ and } \vec{AB} \cdot \vec{OB} = |\vec{AB}| |\vec{OB}| \cos \theta$$

$$\therefore \cos \theta > 0 \Rightarrow \theta \text{ is acute}$$

$$\text{Hence } \theta = 30^\circ$$

(c)

$$\vec{AB} \cdot \vec{OB} = |\vec{AB}| |\vec{OB}| \cos 30^\circ$$

$$\begin{pmatrix} -2 \\ 1 \\ \alpha \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} = \left| \begin{pmatrix} -2 \\ 1 \\ \alpha \end{pmatrix} \right| \left| \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} \right| \cos 30^\circ$$

$$\alpha^2 = \left(\sqrt{5 + \alpha^2} \right) \sqrt{\alpha^2} \left(\frac{\sqrt{3}}{2} \right)$$

$$2\alpha^2 = \sqrt{5 + \alpha^2} \sqrt{\alpha^2} \sqrt{3}$$

$$2\alpha^2 = \sqrt{3\alpha^2(5 + \alpha^2)}$$

$$4\alpha^4 = 3\alpha^2(5 + \alpha^2)$$

$$4\alpha^4 = 15\alpha^2 + 3\alpha^4$$

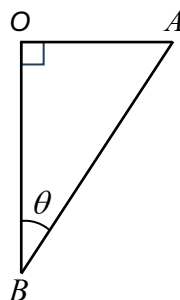
$$\alpha^4 - 15\alpha^2 = 0$$

$$\alpha^2(\alpha^2 - 15) = 0$$

$$\alpha^2 = 0 \text{ (rej. } \because \alpha < 0) \text{ or } \alpha^2 - 15 = 0$$

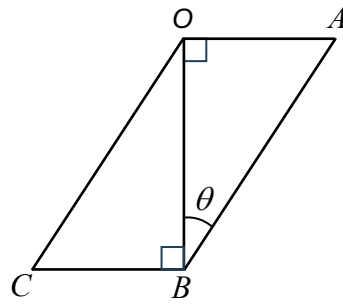
$$\alpha^2 = 15$$

$$\alpha = -\sqrt{15} \text{ (reject } +\sqrt{15} \text{ since } \alpha < 0)$$



(d)

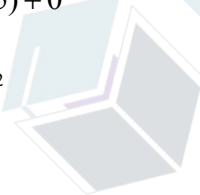
$$\begin{aligned} \vec{c} &= \vec{b} - \vec{a} \\ \vec{OC} &= \vec{OB} - \vec{OA} \\ \vec{OC} &= \vec{AB} \\ \therefore \vec{OC} &\parallel \vec{AB} \end{aligned}$$



$$\begin{aligned} \vec{CB} &= \vec{OB} - \vec{OC} \\ &= \vec{OB} - \vec{AB} \\ &= \vec{OB} - (\vec{OB} - \vec{OA}) \\ &= \vec{OA} \\ \therefore \vec{CB} &\parallel \vec{OA} \end{aligned}$$

Since $\vec{OC} \parallel \vec{AB}$ and $\vec{CB} \parallel \vec{OA}$,
 $OACB$ is a parallelogram.

$$\begin{aligned} |\vec{OA} \times \vec{OC}| &= |\vec{OA} \times \vec{AB}| \\ &= \left| \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ -\sqrt{15} \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} \sqrt{15} \\ 2\sqrt{15} \\ 0 \end{pmatrix} \right| \\ &= \sqrt{15 + 2^2(15) + 0} \\ &= \sqrt{75} \\ &= 5\sqrt{3} \text{ unit}^2 \end{aligned}$$



16

$$(a) \quad \overrightarrow{OP} = \begin{pmatrix} \alpha \\ 1 \\ -1 \end{pmatrix}, \quad \overrightarrow{OQ} = \begin{pmatrix} 3 \\ 0 \\ \alpha-1 \end{pmatrix}, \quad \overrightarrow{PQ} = \begin{pmatrix} 3-\alpha \\ -1 \\ \alpha \end{pmatrix}$$

$$\overrightarrow{PQ} \cdot \begin{pmatrix} 3 \\ 1 \\ \beta \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 3-\alpha \\ -1 \\ \alpha \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ \beta \end{pmatrix} = 0$$

$$\Rightarrow 9 - 3\alpha - 1 + \alpha\beta = 0 \quad \text{--- (1)}$$

$$\overrightarrow{PQ} \cdot \begin{pmatrix} 1 \\ \beta \\ -1 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 3-\alpha \\ -1 \\ \alpha \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \beta \\ -1 \end{pmatrix} = 0$$

$$\Rightarrow 3 - \alpha - \beta - \alpha = 0$$

$$\Rightarrow \beta = 3 - 2\alpha \quad \text{--- (2)}$$

$$8 - 3\alpha + \alpha(3 - 2\alpha) = 0$$

$$2\alpha^2 = 8$$

$$\alpha = \pm 2$$

$\therefore \alpha = 2$ is the only possible solution as $\alpha > 0$.

$$\beta = 3 - 2(2) = -1$$

$$(b) \quad \overrightarrow{PQ} = \begin{pmatrix} 3-2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\text{Shortest distance} = |\overrightarrow{PQ}| = \left| \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right| = \sqrt{6}$$

$$(c) \quad \overrightarrow{PR} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$$

$$\text{A normal to } \Pi \text{ is } \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}$$

$$\mathbf{r} \cdot \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}$$

$$-x + 6y + 3z = 1$$

$$\text{or } x - 6y - 3z = -1$$

(d)

$$\text{Angle required} = 90^\circ - \cos^{-1} \frac{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -6 \\ -3 \end{pmatrix}}{\sqrt{3}\sqrt{46}} = 58.3^\circ$$

OR

$$\text{Angle required} = \sin^{-1} \frac{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -6 \\ -3 \end{pmatrix}}{\sqrt{3}\sqrt{46}} = 58.3^\circ$$

17

(i) $4x + 6y - z = 3$ ---- (1)

$2x + 5y - 2z = 1$ ---- (2)

$5x - 3y + 4z = 9$ ---- (3)

Using G.C., $(2, -1, -1)$.

(ii) $4x + 6y - z = 3$ ---- (1)

$2x + 5y - 2z = 1$ ---- (2)

Using G.C.,

$$x = \frac{9}{8} - \frac{7}{8}z$$

$$y = -\frac{1}{4} + \frac{3}{4}z$$

$$z = z$$

$$\therefore \underline{r} = \begin{pmatrix} \frac{9}{8} \\ -\frac{1}{4} \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{7}{8} \\ \frac{3}{4} \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$$

For the line to lie on the p_3 , line and normal vector are perpendicular, i.e. $\underline{m} \cdot \underline{n} = 0$

$$\begin{pmatrix} -\frac{7}{8} \\ \frac{3}{4} \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ -3 \\ 4 \end{pmatrix} = 0$$

$$-\frac{7}{8}\alpha + \frac{3}{4}(-3) + 4 = 0$$

$$\alpha = 2$$

For the line to lie on the p_3 , a point on the line must lie on the plane, i.e. $\underline{a} \cdot \underline{n} = \underline{r} \cdot \underline{n}$

$$\begin{aligned} \beta &= \begin{pmatrix} \frac{9}{8} \\ -\frac{1}{4} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \\ &= \frac{9}{8}(2) + \left(-\frac{1}{4}\right)(-3) \\ &= 3 \end{aligned}$$

(a) $l_1: \frac{x+1}{4} = \frac{2-y}{3} = z$

$$r = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

$\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ are position vectors of 2 possible points on l_1 , substituting them into p_2 ,

$$\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ 3 \\ \beta \end{pmatrix} = 7$$

$$\begin{aligned} -\alpha + 6 &= 7 \\ \alpha &= -1 \end{aligned}$$

$$\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ 3 \\ \beta \end{pmatrix} = 7$$

$$\begin{aligned} 3\alpha - 3 + \beta &= 7 \\ \beta &= 13 \end{aligned}$$

(b) $n_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$p_1: r \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$

$$x + y - z = 0$$

$p_2: -3x + 3y + 2z = 7$

Using G.C.,

$$x = -\frac{7}{6} + \frac{5}{6}z$$

$$y = \frac{7}{6} + \frac{1}{6}z$$

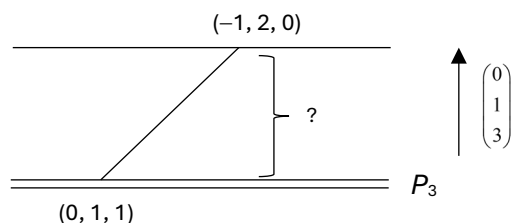
$$z = z$$

$$\therefore r = \frac{7}{6} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}, \mu \in \mathbb{R}.$$

Note that point $(0, -1, -1)$ lies on p_3 .

(c) $\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$

$$\text{Distance} = \frac{\left| \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right|}{\sqrt{1^2 + 3^2}} = \frac{6}{\sqrt{10}}$$



19

(a) Vector equation of the line l is

$$r = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

Let α be the acute angle between the normal vector of plane Π_1 and line l .

$$\begin{aligned} \alpha &= \cos^{-1} \frac{\left| \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \right|}{\left\| \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\| \left\| \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \right\|}} \\ &= \cos^{-1} \left| \frac{3}{\sqrt{6}\sqrt{14}} \right| \\ &= 70.893^\circ \text{ or } 1.2373 \end{aligned}$$

$\therefore$ acute angle between the plane Π_1 and line l is

$$\begin{aligned} &= 90^\circ - 70.893^\circ \text{ or } \frac{\pi}{2} - 1.2373 \\ &= 19.1^\circ \text{ or } 0.333 \end{aligned}$$

(b) Substitute equation of line into the equation of plane:

$$\begin{aligned} \left[\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} &= 3 \\ -3 + 3\lambda &= 3 \\ \lambda &= 2 \end{aligned}$$

When $\lambda = 2$,

$$r = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$$

Coordinates of the point of intersection is (4, 0, 5).

(c) **Method ①:**

Perpendicular distance from B to Π_1

$$\begin{aligned} &= \frac{|\overrightarrow{AB} \cdot \underline{\underline{n}}|}{|\underline{\underline{n}}|} \\ &= \frac{\left| \begin{bmatrix} 10 \\ -4 \\ 7 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right|}{\left| \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right|} \\ &= \frac{\left| \begin{bmatrix} 6 \\ -4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right|}{\sqrt{6}} \\ &= \frac{6}{\sqrt{6}} \\ &= \sqrt{6} \text{ units} \end{aligned}$$

Method ②:

Equation of line that is parallel to normal vector and passing through $(10, -4, 7)$ is:

$$\underline{\underline{r}} = \begin{pmatrix} 10 \\ -4 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mu \in \mathbb{R}$$

At the point of intersection,

$$\begin{aligned} \left[\begin{pmatrix} 10 \\ -4 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} &= 3 \\ 9 + 6\mu &= 3 \\ \mu &= -1 \end{aligned}$$

When $\mu = 1$,

$$\underline{\underline{r}} = \begin{pmatrix} 10 \\ -4 \\ 7 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -5 \\ 8 \end{pmatrix}$$

Perpendicular distance from B to Π_1

$$\begin{aligned}
 &= \left| \begin{pmatrix} 8 \\ -5 \\ 8 \end{pmatrix} - \begin{pmatrix} 10 \\ -4 \\ 7 \end{pmatrix} \right| \\
 &= \left| \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right| \\
 &= \sqrt{(-2)^2 + (-1)^2 + 1^2} \\
 &= \sqrt{6} \text{ units}
 \end{aligned}$$

(d) A vector perpendicular to the plane Π_2 is

$$\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}$$

Equation of plane Π_2 is

$$\begin{aligned}
 \vec{r} \cdot \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} &= \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} \\
 x + 5y + 7z &= 39
 \end{aligned}$$

(e) Note that point $(4, 0, 5)$ lies on both planes Π_1 and Π_2 .

A vector parallel to the line of intersection of both planes is

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 12 \\ -15 \\ 9 \end{pmatrix} = 3 \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$$

Vector equation of the line that lies in both planes is

$$\vec{r} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}, \mu \in \mathbb{R}$$

20

- (i) Note that $A(4, 0, 0)$ lies on p .

Perpendicular distance from the origin to p

$$\begin{aligned} & \left| \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} \right| \\ &= \frac{\left| \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} \right|}{\sqrt{1^2 + 2^2 + (-5)^2}} \\ &= \frac{4}{\sqrt{30}} \text{ units} \end{aligned}$$

- (ii) l lies on p means l is parallel to p and a point on l lies on p .

l is parallel to p means direction vector of l is perpendicular to normal of p :

$$\begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} a \\ 2a \\ 2 \end{pmatrix} = 0$$

$$a + 4a - 10 = 0$$

$$a = 2 \text{ (shown)}$$

We need $\begin{pmatrix} 3 \\ -2 \\ b \end{pmatrix}$ to lie on p : $\begin{pmatrix} 3 \\ -2 \\ b \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} = 4$

$$3 - 4 - 5b = 4$$

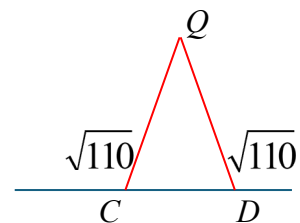
$$5b = -5$$

$$b = -1 \text{ (shown)}$$

(iii)

Let $\overrightarrow{OR} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$ be the position vector of a point on l for some $\lambda \in \mathbb{R}$.

$$\overrightarrow{QR} = \left[\begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \right] - \begin{pmatrix} -4 \\ 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 7+2\lambda \\ -9+4\lambda \\ -4+2\lambda \end{pmatrix}$$



$$|\overrightarrow{QR}| = \sqrt{110}$$

$$\sqrt{(7+2\lambda)^2 + (-9+4\lambda)^2 + (-4+2\lambda)^2} = \sqrt{110}$$

$$49 + 28\lambda + 4\lambda^2 + 81 - 72\lambda + 16\lambda^2 + 16 - 16\lambda + 4\lambda^2 = 110$$

$$24\lambda^2 - 60\lambda + 36 = 0$$

$$2\lambda^2 - 5\lambda + 3 = 0$$

$$(2\lambda - 3)(\lambda - 1) = 0$$

$$(2\lambda - 3) = 0 \text{ or } (\lambda - 1) = 0$$

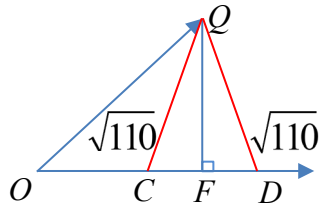
$$\lambda = \frac{3}{2} \text{ or } \lambda = 1$$

$$\text{When } \lambda = \frac{3}{2}, \overrightarrow{OC} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$$

$$\text{When } \lambda = 1, \overrightarrow{OD} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

- (iv) Let F be the foot of perpendicular from Q to l . Then QC and QD are the hypotenuses of the right-angled triangles QFC and QFD respectively.

Therefore the shortest distance from Q to l , QF , has to be smaller than QC and QD , which is $\sqrt{110}$.



OR

Assume the shortest distance from Q to l is greater than $\sqrt{110}$. But there are points C and D found on l that are a distance of $\sqrt{110}$ away from Q , which contradicts the original assumption. Therefore the shortest distance from Q to l , has to be smaller than $\sqrt{110}$.

OR

There should only be one unique point on l , the foot of perpendicular, that is the shortest distance from Q . Currently there are 2 points C and D found on l that are a distance of $\sqrt{110}$ away from Q . As the distance is shortened, C and D will converge to the foot of perpendicular.



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(a)(i) $|\underline{a} \times \underline{b}|$ is the area of parallelogram with adjacent sides OA and OB

OR

$|\underline{a} \times \underline{b}|$ is twice the area of triangle OAB

(a)(ii) Since $\underline{c}$ is perpendicular to $\underline{a} \times \underline{b}$, $\underline{c}$ which denotes $\overrightarrow{OC}$

(1) lies on the plane containing O and

(2) parallel to $\underline{a}$ and $\underline{b}$.

Hence $\underline{c}$ can be written as $\underline{c} = \mu \underline{a} + \lambda \underline{b}$.

OR

Since $\underline{c}$ is perpendicular to $\underline{a} \times \underline{b}$, it denotes $\underline{c}$ is

(1) parallel to $\underline{a}$ and $\underline{b}$ and

(2) since $\underline{a}$, $\underline{b}$ and $\underline{c}$ are position vector containing same point O ,

(3) hence $\underline{c}$ will lie on the same plane as $\underline{a}$ and $\underline{b}$ (OR point O, A, B and C lies on the same plane).

Therefore $\underline{c}$ can be written as $\underline{c} = \mu \underline{a} + \lambda \underline{b}$.

(a)(iii) Since angle $AOC = \frac{\pi}{2}$, we have

$$\underline{c} \cdot \underline{a} = 0$$

$$(\mu \underline{a} + \lambda \underline{b}) \cdot \underline{a} = 0$$

$$\mu \underline{a} \cdot \underline{a} + \lambda \underline{b} \cdot \underline{a} = 0$$

$$\mu |\underline{a}|^2 + \lambda |\underline{b}| |\underline{a}| \cos \frac{\pi}{3} = 0$$

$$\mu (1)^2 + \lambda (2)(1) \left(\frac{1}{2} \right) = 0$$

$$\mu = -\lambda$$

Hence $\underline{c} = \mu \underline{a} + \lambda \underline{b}$

$$\underline{c} = -\lambda \underline{a} + \lambda \underline{b}$$

$$\text{Given } |\underline{c}| = \sqrt{3} \Rightarrow |-\lambda \underline{a} + \lambda \underline{b}|^2 = 3$$

$$\lambda^2 |-\underline{a} + \underline{b}|^2 = 3$$

$$\lambda^2 (-\underline{a} + \underline{b}) \cdot (-\underline{a} + \underline{b}) = 3$$

$$\lambda^2 (\underline{a} \cdot \underline{a} - 2\underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{b}) = 3$$

$$\lambda^2 \left(|\underline{a}|^2 - 2|\underline{a}| |\underline{b}| \cos \frac{\pi}{3} + |\underline{b}|^2 \right) = 3$$

$$\lambda^2 (1 - 2 + 2^2) = 3$$

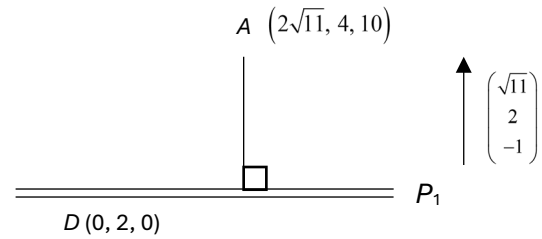
$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

(b)(i) Let $D(0, 2, 0)$ be a point on plane p_1

$$\overrightarrow{AD} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 2\sqrt{11} \\ 4 \\ 10 \end{pmatrix} = \begin{pmatrix} -2\sqrt{11} \\ -2 \\ -10 \end{pmatrix}$$

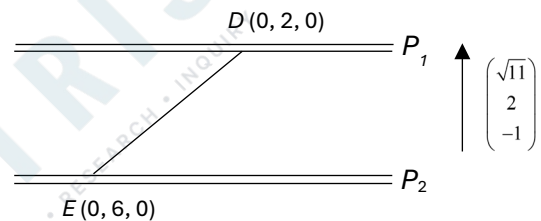
$$\begin{aligned} \text{Distance of } A \text{ from } p_1 &= \frac{\left| \begin{pmatrix} -2\sqrt{11} \\ -2 \\ -10 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{11} \\ 2 \\ -1 \end{pmatrix} \right|}{\sqrt{(\sqrt{11})^2 + 2^2 + (-1)^2}} \\ &= \frac{22 + 4 - 10}{4} \\ &= 4 \end{aligned}$$



(b)(ii) Let $E(0, 6, 0)$ be a point on plane p_2

$$\overrightarrow{DE} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \text{Distance between 2 planes} &= \frac{\left| \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{11} \\ 2 \\ -1 \end{pmatrix} \right|}{\sqrt{(\sqrt{11})^2 + 2^2 + (-1)^2}} = \frac{8}{4} = 2 \end{aligned}$$



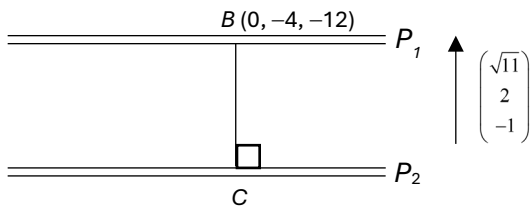
Since the distance of A from p_1 is larger than the distance between the 2 planes, therefore the point A will not lie between the 2 planes.

- (b)(iii)** For the line BC to be perpendicular to both p_1 and p_2 , C will be the foot of perpendicular of B onto p_2 .

Since BC is parallel to normal of p_2 ,

$$\therefore \overrightarrow{BC} = k \begin{pmatrix} \sqrt{11} \\ 2 \\ -1 \end{pmatrix}$$

$$\Rightarrow \overrightarrow{OC} = \begin{pmatrix} 0 \\ -4 \\ -12 \end{pmatrix} + k \begin{pmatrix} \sqrt{11} \\ 2 \\ -1 \end{pmatrix}$$



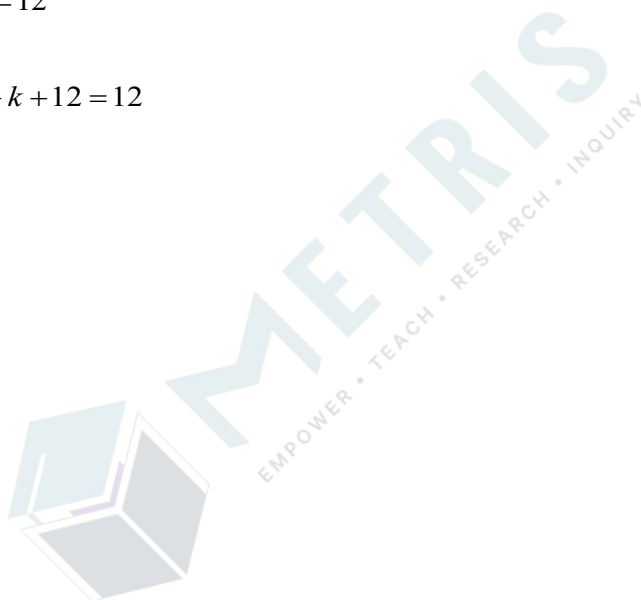
Since C is a point on p_2 , we have

$$\begin{pmatrix} \sqrt{11}k \\ 2k-4 \\ -k-12 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{11} \\ 2 \\ -1 \end{pmatrix} = 12$$

$$\Rightarrow 11k + 4k - 8 + k + 12 = 12$$

$$\Rightarrow k = \frac{1}{2}$$

$$\therefore \overrightarrow{OC} = \begin{pmatrix} \frac{\sqrt{11}}{2} \\ -3 \\ -\frac{25}{2} \end{pmatrix}$$

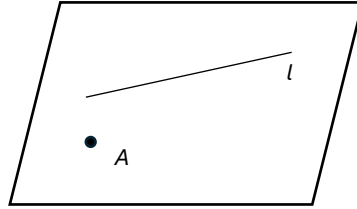


(i)

$$l_1: \vec{r} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R}$$

Let $\vec{OB} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$. Given that $\vec{OA} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$

$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$



Let $\vec{n}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$$\pi_1: \vec{r} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{r} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 3$$

$$x - y - z = -3$$



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(ii) Q is the foot of perpendicular of P on l_1

$$\overrightarrow{OQ} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ for some } \lambda$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$$

$$\overrightarrow{PQ} = \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\overrightarrow{PQ}$ perpendicular to the line l_1 ,

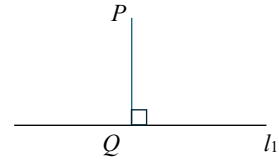
$$\overrightarrow{PQ} \cdot \vec{d} = 0$$

$$\left[\begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$-1 + 2\lambda = 0$$

$$\lambda = \frac{1}{2}$$

$$\overrightarrow{OQ} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \\ 3 \end{pmatrix}$$



(iii) l_2 parallel to $\pi_1 \Rightarrow \vec{d}_2$ is perpendicular to $\vec{n}_1$

$$\vec{d}_2 \cdot \vec{n}_1 = 0$$

$$\begin{pmatrix} 3 \\ 1 \\ m \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$-3 + 1 + m = 0$$

$$m = 2$$

h = distance between l_2 and π_1

$$h = \text{distance between } P(-3, 4, 5) \text{ and } \pi_1 : \vec{r} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 3$$

Let N be the foot of perpendicular of P on π_1 .

$$l_{PN} : \vec{r} = \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R} \quad \text{--- (1)}$$

$$\pi_1 : \vec{r} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 3 \quad \text{--- (2)}$$

To find N , sub (1) into (2):

$$\begin{aligned} \left[\begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right] \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} &= 3 \\ \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} &= 3 \\ 12 + 3\lambda &= 3 \\ \lambda &= -3 \end{aligned}$$

$$\overrightarrow{ON} = \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$h = \overrightarrow{PN} = \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} = -3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$PN = \left| -3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right| = 3 \left| \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right| = 3\sqrt{3}$$

(iv) $PQ:PR = 2:5$

Case ①: Q is between P and R

By similar triangles,

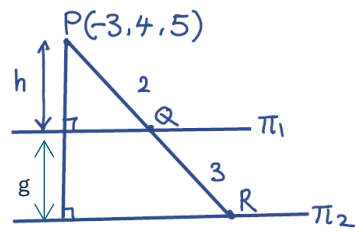
$$\frac{2}{h} = \frac{5}{g+h}$$

$$2g + 2h = 5h$$

$$2g = 3h$$

$$g = \frac{3h}{2}$$

Distance between π_1 and $\pi_2 = \frac{3h}{2}$



Case ②: P is between Q and R

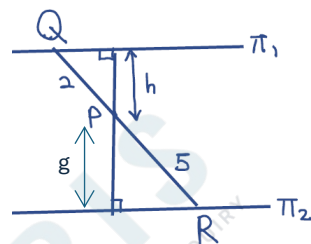
By similar triangles,

$$\frac{2}{h} = \frac{5}{g}$$

$$2g = 5h$$

$$g = \frac{5h}{2}$$

Distance between π_1 and $\pi_2 = \frac{5h}{2} + h = \frac{7h}{2}$



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(a)

$$l_{AB} : \vec{r} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ a \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

$$l_{AC} : \vec{r} = \begin{pmatrix} 4 \\ 4 \\ 9 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ b \\ 2 \end{pmatrix}, \mu \in \mathbb{R}$$

Since the lines AB and AC meet, $\begin{pmatrix} 1+2\lambda \\ 4+a\lambda \\ 6+\lambda \end{pmatrix} = \begin{pmatrix} 4+\mu \\ 4+b\mu \\ 9+2\mu \end{pmatrix}.$

$$2\lambda - \mu = 3 \quad \text{--- (1)}$$

$$a\lambda = b\mu \quad \text{--- (2)}$$

$$\lambda - 2\mu = 3 \quad \text{--- (3)}$$

Solving (1) & (3) gives $\lambda = 1$ & $\mu = -1$

Hence (2) gives $a = -b \Rightarrow a + b = 0$ (shown)

(b)

$$\begin{pmatrix} 2 \\ a \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ b \\ 2 \end{pmatrix} = \begin{pmatrix} 2a-b \\ -3 \\ 2b-a \end{pmatrix}$$

Since the normal of plane ABC is parallel to $\begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2a-b \\ -3 \\ 2b-a \end{pmatrix} = k \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}$

By observation, $k = -3$

$$2a - b = 9 \quad \text{--- (1)}$$

$$2b - a = -9 \quad \text{--- (2)}$$

Solving (1) & (2) gives $a = 3$ & $b = -3$

Equation of plane ABC : $\vec{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}$

$$\vec{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} = 19$$

$$-3x + y + 3z = 19$$

- (c) Since $\lambda = 1$ & $\mu = -1$ and $a = 3$ & $b = -3$, the coordinates of A are $(3, 7, 7)$.

Let the acute angle between lines AB and AC be θ .

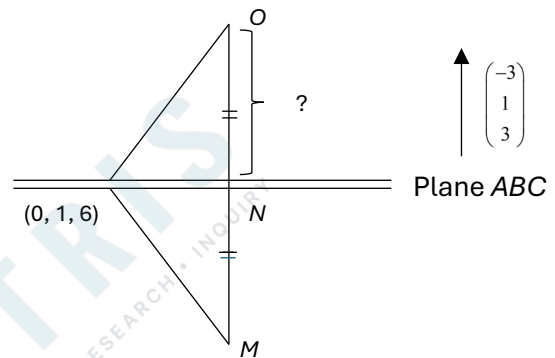
$$\cos \theta = \frac{\left| \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \right|}{\sqrt{4+9+1}\sqrt{1+9+4}} = \frac{5}{14}$$

$$\theta = 69.1^\circ \text{ or } 1.21 \text{ rad}$$

- (d) Note that point $(0, 1, 6)$ lies on plane ABC .

Distance of plane ABC from the origin

$$\begin{aligned} & \frac{\left| \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} \right|}{\sqrt{9+1+9}} \\ &= \frac{19}{\sqrt{19}} \\ &= \sqrt{19} \text{ units} \end{aligned}$$



$$l_{ON} : \vec{r} = s \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}, s \in \mathbb{R}$$

$$\text{Plane } ABC: \vec{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} = 19$$

$$\begin{pmatrix} -3s \\ s \\ 3s \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} = 19$$

$$9s + s + 9s = 19$$

$$s = 1$$

$$\therefore \vec{ON} = \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}$$

$$\vec{OM} = 2 \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 6 \end{pmatrix}$$

Hence the two required planes:

$$\begin{aligned} \vec{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} & \vec{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} &= \begin{pmatrix} -6 \\ 2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} \\ & & & \\ \vec{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} &= 0 & \vec{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} &= 18 + 2 + 18 \\ & & & \\ & & \vec{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} &= 38 \end{aligned}$$



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(a)

Since p and q are perpendicular, a direction vector parallel to q is $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.

$$\text{Normal to } q = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -4 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\text{Equation of } q: \vec{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\vec{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 7$$

$$x + 2y + 2z = 7$$

(b) $2x - z = 10$ --- (1)

$x + 2y + 2z = 7$ --- (2)

Using G.C.:

$$x = 5 + \frac{1}{2}z$$

$$y = 1 - \frac{5}{4}z$$

$$z = z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{4}z \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}$$

Let $\mu = \frac{1}{4}z$, equation of m is $\vec{r} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}, \mu \in \mathbb{R}$

- (c) Let F be the foot of perpendicular from A to q .

$$l_{AF}: \underline{r} = \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \alpha \in \mathbb{R}$$

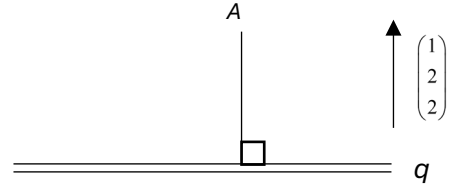
Since F lies on l_{AF} , $\overrightarrow{OF} = \begin{pmatrix} 2 + \alpha \\ 1 + 2\alpha \\ -6 + 2\alpha \end{pmatrix}$ for some $\alpha \in \mathbb{R}$

Since F also lies on q , $(2 + \alpha) + 2(1 + 2\alpha) + 2(-6 + 2\alpha) = 7$
 $-8 + 9\alpha = 7$

$$\alpha = \frac{5}{3}$$

$$\therefore \overrightarrow{OF} = \begin{pmatrix} 11/3 \\ 13/3 \\ -8/3 \end{pmatrix}$$

$$k = |\overrightarrow{AF}| = \left| \begin{pmatrix} 5/3 \\ 10/3 \\ 10/3 \end{pmatrix} \right| = 5$$



- (d) The other point lying on p with shortest distance 5 units from q is A' , the reflection of A about m .

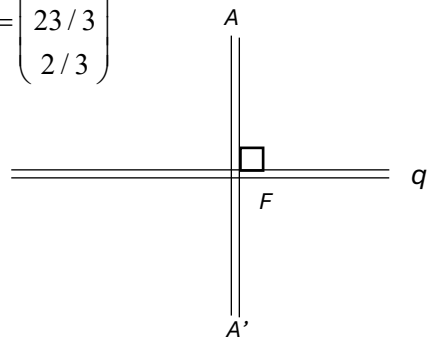
By ratio theorem, $\overrightarrow{OF} = \frac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$

$$\overrightarrow{OA'} = 2\overrightarrow{OF} - \overrightarrow{OA} = 2 \begin{pmatrix} 11/3 \\ 13/3 \\ -8/3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix} = \begin{pmatrix} 16/3 \\ 23/3 \\ 2/3 \end{pmatrix}$$

Both the lines must be parallel to m .

$$\therefore \text{Equation of lines: } \underline{r} = \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}, \quad \alpha \in \mathbb{R}$$

$$\text{and } \underline{r} = \begin{pmatrix} 16/3 \\ 23/3 \\ 2/3 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}, \quad \beta \in \mathbb{R}$$

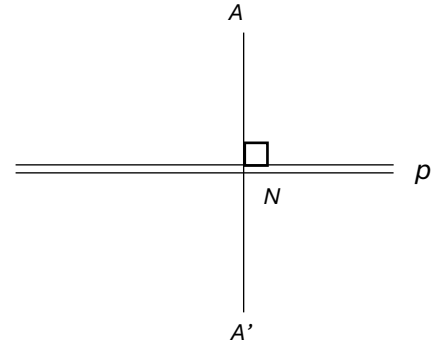


(i)

Given $p: \mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 12$.

Equation of the line $AN: \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ where $\lambda \in \mathbb{R}$.

Then $\overrightarrow{ON} = \begin{pmatrix} 2+\lambda \\ 1-2\lambda \\ -3+2\lambda \end{pmatrix}$ for a specific value of λ .



Since the point N lies on the plane p ,

$$\begin{pmatrix} 2+\lambda \\ 1-2\lambda \\ -3+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 12$$

$$2 + \lambda - 2(1 - 2\lambda) + 2(-3 + 2\lambda) = 12$$

$$\lambda = 2$$

Thus $\overrightarrow{ON} = \begin{pmatrix} 2+2 \\ 1-4 \\ -3+4 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$. The coordinates of the point N is $(4, -3, 1)$

Let A' be the point of reflection of A .

By ratio theorem, $\overrightarrow{ON} = \frac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$

$$\overrightarrow{OA'} = 2\overrightarrow{ON} - \overrightarrow{OA}$$

$$= 2 \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

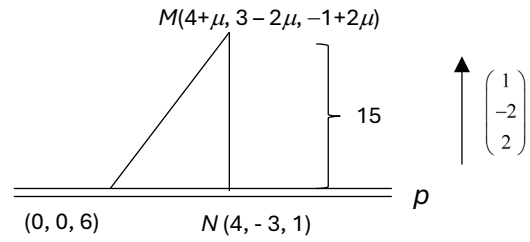
$$= \begin{pmatrix} 6 \\ -7 \\ 5 \end{pmatrix}$$

The coordinates of the reflection of A in p is $(6, -7, 5)$.

(ii)

$$l_{MN} : \vec{r} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \mu \in \mathbb{R}$$

$$\therefore \overrightarrow{OM} = \begin{pmatrix} 4 + \mu \\ -3 - 2\mu \\ 1 + 2\mu \end{pmatrix} \text{ for some } \mu$$



Note that point $(0, 0, 6)$ lies on p .

$$\begin{pmatrix} 4 + \mu \\ -3 - 2\mu \\ 1 + 2\mu \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 + \mu \\ -3 - 2\mu \\ -5 + 2\mu \end{pmatrix}$$

$$\frac{\left| \begin{pmatrix} 4 + \mu \\ -3 - 2\mu \\ -5 + 2\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \right|}{\sqrt{1^2 + 2^2 + 2^2}} = 15$$

$$|4 + \mu - 2(-3 - 2\mu) + 2(-5 + 2\mu)| = 45$$

$$|4 + \mu + 6 + 4\mu - 10 + 4\mu| = 45$$

$$|9\mu| = 45$$

$$\mu = \pm 5$$

$$\overrightarrow{OM} = \begin{pmatrix} 4 + 5 \\ -3 - 2(5) \\ 1 + 2(5) \end{pmatrix} = \begin{pmatrix} 9 \\ -13 \\ 11 \end{pmatrix} \quad \text{or} \quad \overrightarrow{OM} = \begin{pmatrix} 4 + (-5) \\ -3 - 2(-5) \\ 1 + 2(-5) \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -9 \end{pmatrix}$$

Equations of planes:

$$\vec{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ -13 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$\vec{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 9 + 26 + 22$$

$$\vec{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 57$$

$$x - 2y + 2z = 57$$

or

$$\vec{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -9 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$\vec{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = -1 - 14 - 18$$

$$\vec{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = -33$$

$$x - 2y + 2z = -33$$

(a) If $\vec{n}$ is perpendicular to $\vec{a}$, $\vec{n} \cdot \vec{a} = 0$.

$$\begin{aligned} & [(\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b}] \cdot \vec{a} \\ &= (\vec{a} \cdot \vec{b})\vec{a} \cdot \vec{a} - (\vec{a} \cdot \vec{a})\vec{b} \cdot \vec{a} \\ &= 0 \quad \text{since } \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \text{ (shown)} \end{aligned}$$

If $\vec{n}$ is parallel to plane OAB , it is perpendicular to $\vec{a} \times \vec{b}$ since $\vec{a} \times \vec{b}$ is perpendicular to plane OAB .

i.e. $\vec{n} \cdot (\vec{a} \times \vec{b}) = 0$.

$$\begin{aligned} & [(\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b}] \cdot (\vec{a} \times \vec{b}) \\ &= (\vec{a} \cdot \vec{b})\vec{a} \cdot (\vec{a} \times \vec{b}) - (\vec{a} \cdot \vec{a})\vec{b} \cdot (\vec{a} \times \vec{b}) \\ &= 0 \quad \text{since } \vec{a} \perp \vec{a} \times \vec{b} \text{ and } \vec{b} \perp \vec{a} \times \vec{b} \text{ (shown)} \end{aligned}$$

Since $\vec{a} \perp \vec{a} \times \vec{b}$, thus $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$
Also $\vec{b} \perp \vec{a} \times \vec{b}$, thus $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$

OR

Since $\vec{n}$ is a linear combination of non-zero and non-parallel vectors $\vec{a}$ and $\vec{b}$, $\vec{n}$ lies on the same plane as $\vec{a}$ and $\vec{b}$ ie plane OAB . Hence $\vec{n}$ is parallel to the plane OAB .

(b) $\vec{n} = (\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b}$

$$\begin{aligned} &= \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right] \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \\ &= 6 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \\ \vec{m} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \text{or} \quad \vec{m} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \end{aligned}$$

(c)

$$\text{Line } OA : \underline{r} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

Let l be the line which passes through point B and is parallel to $\underline{m}$.

$$l : \underline{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \mu \in \mathbb{R}$$

When line OA intersects l ,

$$\lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\lambda = 2$$

$$\lambda = 1 + \mu \Rightarrow \mu = 1$$

$$\lambda = 3 - \mu \Rightarrow \mu = 1$$

$\therefore$ Coordinates of point of intersection are $(2, 2, 2)$.

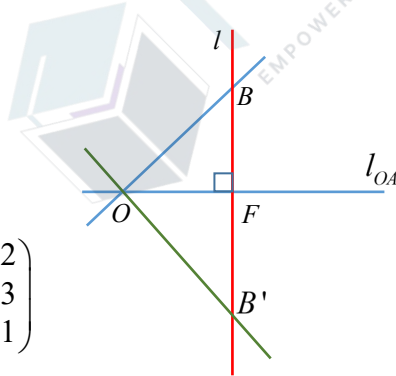
(d) Note that $\underline{m} \parallel \underline{n}$, and $\underline{n}$ is perpendicular to $\underline{a}$.
Thus, line l is perpendicular to line OA .

By Ratio Theorem,

$$\overrightarrow{OF} = \frac{\overrightarrow{OB} + \overrightarrow{OB'}}{2}$$

$$\overrightarrow{OB'} = 2\overrightarrow{OF} - \overrightarrow{OB}$$

$$= 2 \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$



$$\text{Equation of line of reflection is } \underline{r} = \beta \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \beta \in \mathbb{R}$$

$\therefore$ A cartesian equation of the line of reflection is $\frac{x}{2} = \frac{y}{3} = z$.

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(a) $\overrightarrow{AB} = \begin{pmatrix} -9 \\ 2 \\ 0 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -9 \\ 0 \\ 3 \end{pmatrix}$.

$$\begin{pmatrix} -9 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -9 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 27 \\ 18 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 9 \\ 6 \end{pmatrix}$$

$$\vec{r} \cdot \begin{pmatrix} 2 \\ 9 \\ 6 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 9 \\ 6 \end{pmatrix}$$

$$\vec{r} \cdot \begin{pmatrix} 2 \\ 9 \\ 6 \end{pmatrix} = 18$$

$$2x + 9y + 6z = 18$$

(b)

$$\begin{aligned} \text{The required angle} &= \sin^{-1} \frac{\left| \begin{pmatrix} 2 \\ 9 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|}{\sqrt{2^2 + 9^2 + 6^2} \times 1} \\ &= \sin^{-1} \frac{6}{11} \\ &= 33.1^\circ \text{ (1 d.p.)} \end{aligned}$$

(c)

Let the foot of perpendicular be N . Then $\overrightarrow{ON} = \lambda \begin{pmatrix} 2 \\ 9 \\ 6 \end{pmatrix}$.

Substitute $(2\lambda, 9\lambda, 6\lambda)$ to $2x + 9y + 6z = 18$,

$$4\lambda + 81\lambda + 36\lambda = 18$$

$$121\lambda = 18$$

$$\lambda = \frac{18}{121}$$

N has coordinates $\left(\frac{36}{121}, \frac{162}{121}, \frac{108}{121} \right)$.

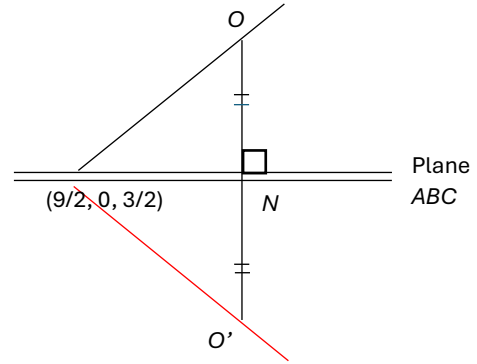
(d) The midpoint of AC is $\left(\frac{9}{2}, 0, \frac{3}{2}\right)$.

The reflection of O in ABC is $\left(\frac{72}{121}, \frac{324}{121}, \frac{216}{121}\right)$.

For direction vector for the reflection line:

$$\begin{pmatrix} \frac{9}{2} \\ 0 \\ \frac{3}{2} \end{pmatrix} - \begin{pmatrix} \frac{72}{121} \\ \frac{324}{121} \\ \frac{216}{121} \end{pmatrix} = \begin{pmatrix} \frac{945}{242} \\ -\frac{324}{121} \\ -\frac{69}{242} \end{pmatrix} = \frac{3}{242} \begin{pmatrix} 315 \\ -216 \\ -23 \end{pmatrix}.$$

A vector equation is $\mathbf{r} = \begin{pmatrix} \frac{9}{2} \\ 0 \\ \frac{3}{2} \end{pmatrix} + \mu \begin{pmatrix} 315 \\ -216 \\ -23 \end{pmatrix}, \mu \in \mathbb{R}.$



(i)

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -4 \\ 0 \\ -2 \end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 8 \\ 8 \\ -16 \end{pmatrix} = 8 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Take the normal vector to the plane as $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

Equation of the plane $ABDC$ is given by

$$\vec{r} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = -4$$

The cartesian equation of the plane $ABDC$ is $x + y - 2z = -4$.

(ii) Let the acute angle be θ .

$$\cos \theta = \frac{\left| \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|}{\sqrt{6} \cdot 1} = \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}$$

$$\theta = 35.3^\circ \text{ (1 dec place) or } 0.615 \text{ radians}$$

(iii) Since D lies on plane $ABDC$, from **part (i)**

$$x + y - 2z = -4$$

Substitute $D(-2, 4, k)$,

$$-2 + 4 - 2k = -4$$

$$-2k = -6$$

$$k = 3$$

(iv)

$$\overrightarrow{BD} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$$

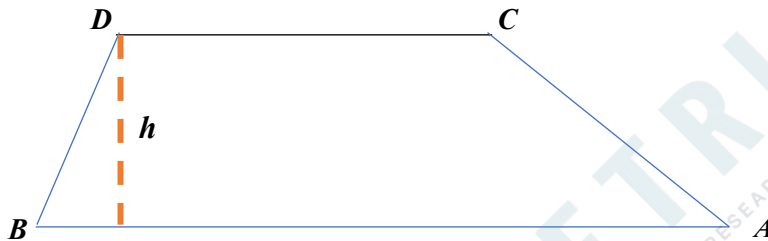
Since $\overrightarrow{BD}$ cannot be expressed as $\overrightarrow{BD} = k\overrightarrow{AC}$, where k is a constant, hence $\overrightarrow{BD}$ and $\overrightarrow{AC}$ are NOT parallel.

$$\overrightarrow{AB} = \begin{pmatrix} -4 \\ 0 \\ -2 \end{pmatrix}, \overrightarrow{CD} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AB} = 2\overrightarrow{CD}$$

Hence $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel.

Since $ABDC$ has one pair of parallel sides and one pair of non-parallel sides, hence $ABDC$ is a trapezium.



Method ①:

To find height of the trapezium $ABDC$, we use

$$\begin{aligned} h &= \frac{\left| \overrightarrow{BD} \times \overrightarrow{BA} \right|}{\left| \overrightarrow{BA} \right|} \\ &= \frac{\left| \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} \times \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} \right|}{\sqrt{20}} \\ &= \frac{\left| \frac{1}{\sqrt{20}} \begin{pmatrix} 8 \\ 8 \\ -16 \end{pmatrix} \right|}{1} \\ &= \frac{4}{5} \sqrt{30} \end{aligned}$$

$$CD = \sqrt{5}$$

$$AB = 2\sqrt{5}$$

Area of trapezium $ABDC$

$$\begin{aligned} &= \frac{1}{2} (CD + AB) h \\ &= \frac{1}{2} (3\sqrt{5}) \left(\frac{4\sqrt{30}}{5} \right) \\ &= 6\sqrt{6} \\ &= 6^{\frac{3}{2}} \text{ units}^2 \end{aligned}$$

$$\therefore a = 6$$

Method ②:

Trapezium $ABDC$

= Area of triangle ABD + Area of triangle ACD

$$= \frac{1}{2} |\overrightarrow{BD} \times \overrightarrow{BA}| + \frac{1}{2} |\overrightarrow{CD} \times \overrightarrow{CA}|$$

$$= \frac{1}{2} |\overrightarrow{BD} \times \overrightarrow{BA}| + \frac{1}{2} |\overrightarrow{CD} \times \overrightarrow{CA}|$$

$$= \frac{1}{2} \left| \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} \times \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} \right| + \frac{1}{2} \left| \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix} \right|$$

$$= \frac{1}{2} \left| \begin{pmatrix} 8 \\ 8 \\ -16 \end{pmatrix} \right| + \frac{1}{2} \left| \begin{pmatrix} -4 \\ -4 \\ 8 \end{pmatrix} \right|$$

$$= 4\sqrt{1^2 + 1^2 + (-2)^2} + 2\sqrt{1^2 + 1^2 + (-2)^2}$$

$$= 6\sqrt{6}$$

$$= 6^{\frac{3}{2}} \text{ units}^2$$

$$\therefore a = 6$$



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(a) $\vec{OA} = \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix}, \vec{OB} = \begin{pmatrix} 8 \\ -4 \\ 3 \end{pmatrix}$

Since F lies on l_1 ,

$$\vec{OF} = \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R}.$$

To find point F ,

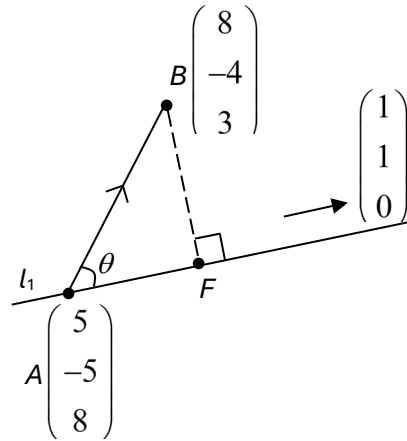
$$\left[\begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 8 \\ -4 \\ 3 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\begin{pmatrix} -3 + \lambda \\ -1 + \lambda \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$-3 + \lambda - 1 + \lambda = 0$$

$$\lambda = 2$$

$$\vec{OF} = \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ 8 \end{pmatrix}$$



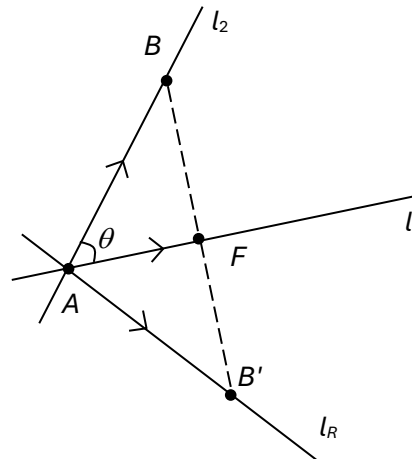
(b) $\vec{OF} = \frac{\vec{OB} + \vec{OB'}}{2}$

$$\vec{OB'} = 2\vec{OF} - \vec{OB}$$

$$= 2 \begin{pmatrix} 7 \\ -3 \\ 8 \end{pmatrix} - \begin{pmatrix} 8 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 13 \end{pmatrix}$$

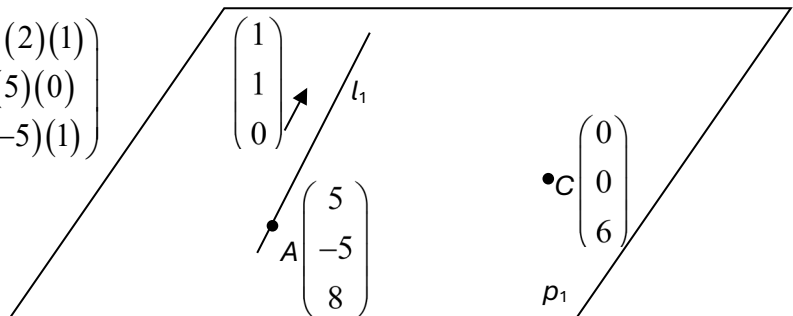
$$\vec{AB'} = \begin{pmatrix} 6 \\ -2 \\ 13 \end{pmatrix} - \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

$$l_R : \vec{r} = \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \mu \in \mathbb{R}$$



$$(c) \quad \overrightarrow{CA} = \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \vec{n}_1 &= \begin{pmatrix} 5 \\ -5 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} (-5)(0) - (2)(1) \\ (2)(1) - (5)(0) \\ (5)(1) - (-5)(1) \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 2 \\ 10 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} \end{aligned}$$



$$\vec{r} \cdot \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} = 30$$

$$p_1: -x + y + 5z = 30 \text{ (shown)}$$

$$(d) \quad p_1: -x + y + 5z = 30 \quad \text{--- (1)}$$

$$p_2: -17x - 37y + 4z = 24 \quad \text{--- (2)}$$

Using G.C.,

$$x = -21 + \frac{7}{2}z$$

$$y = 9 - \frac{3}{2}z$$

$$z = z$$

$$l_s: \vec{r} = \begin{pmatrix} -21 \\ 9 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} \frac{7}{2} \\ -\frac{3}{2} \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}$$

$$(e) \quad p_3: -3x + \alpha y + 15z = \beta$$

$$p_3: \vec{r} \cdot \begin{pmatrix} -3 \\ \alpha \\ 15 \end{pmatrix} = \beta$$

Since $p_1 \parallel p_3$,

$$\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} = k \begin{pmatrix} -3 \\ \alpha \\ 15 \end{pmatrix}$$

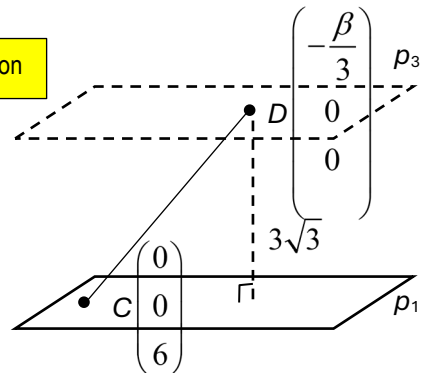
$$k = \frac{1}{3}, \alpha = 3$$

Let a point on p_3 be D .

$$\overrightarrow{OD} = \begin{pmatrix} \frac{\beta}{3} \\ 0 \\ 0 \end{pmatrix}$$

Point D is obtained by trial and error/ observation

$$\overrightarrow{CD} = \begin{pmatrix} \frac{\beta}{3} \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{\beta}{3} \\ 0 \\ -6 \end{pmatrix}$$



Length of projection = $3\sqrt{3}$

$$\frac{\left| \begin{pmatrix} \frac{\beta}{3} \\ 0 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} \right|}{\left| \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} \right|} = 3\sqrt{3}$$

$$\left| -\frac{\beta}{3} + 30 \right| = 3\sqrt{3}(\sqrt{27})$$

$$-\frac{\beta}{3} + 30 = \pm 27$$

$$-\frac{\beta}{3} = -57 \quad \text{or} \quad -\frac{\beta}{3} = -3$$

$$\beta = 171 \quad \text{or} \quad \beta = 9$$

- (a) Let A, B and C be the points $(6, 9, 3)$, $(-2, 13, 1)$ and $(4, 10, 0)$.

$$\overrightarrow{AB} = \begin{pmatrix} -2 \\ 13 \\ 1 \end{pmatrix} - \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} = \begin{pmatrix} -8 \\ 4 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 4 \\ 10 \\ 0 \end{pmatrix} - \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$$

$$\vec{n} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 6-1 \\ -(-12-(-2)) \\ 4-4 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 0 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\pi : \vec{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 24$$

$$x + 2y = 24 \text{ (shown)}$$

- (b) Let l represent the path of the laser beam.

$$\vec{d} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$l : \vec{r} = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

Let θ be the angle between the laser beam and the reflective shield.

$$\begin{aligned} \theta &= \sin^{-1} \frac{\left| \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right|}{\sqrt{2^2 + 2^2 + 1^2} \sqrt{1^2 + 2^2}} \\ &= 63.4^\circ \text{ (1 d.p.)} \end{aligned}$$

(c) Let P be the point of intersection between the laser beam and reflective shield.

$$\left[\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 24$$

$$2\lambda - 6 + 4\lambda = 24$$

$$6\lambda = 30$$

$$\lambda = 5$$

$$\overrightarrow{OP} = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} + 5 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \\ 6 \end{pmatrix}$$

(d) Let Q be the point $(0, -3, 1)$ and the foot of perpendicular from Q to the reflective shield be N .

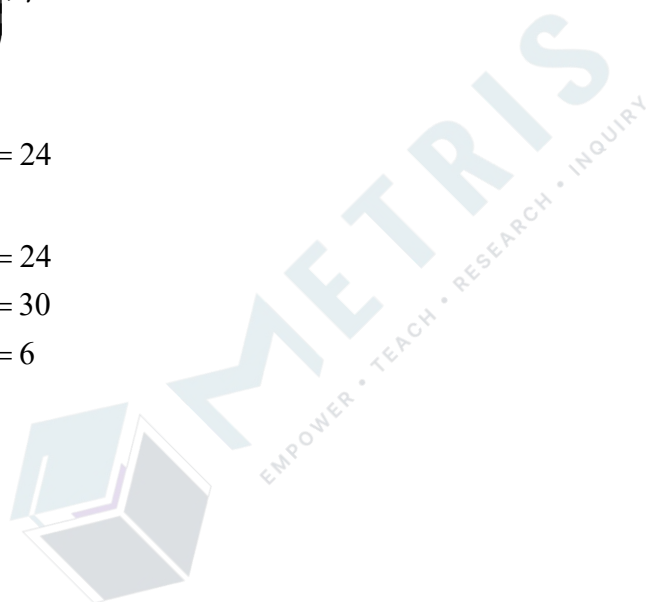
$$l_{QN} : \vec{r} = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \mu \in \mathbb{R}$$

$$\left[\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 24$$

$$\mu - 6 + 4\mu = 24$$

$$5\mu = 30$$

$$\mu = 6$$



$$\overrightarrow{ON} = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} + 6 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 1 \end{pmatrix}$$

Let Q' be the reflected point of Q on the shield.

Using mid-point theorem,

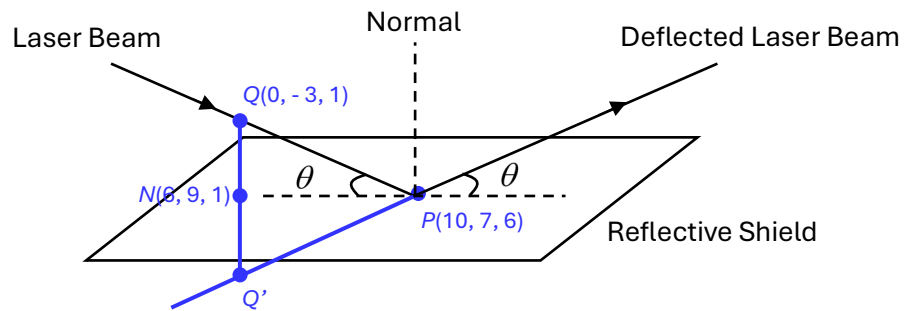
$$\overrightarrow{ON} = \frac{\overrightarrow{OQ} + \overrightarrow{OQ'}}{2}$$

$$\begin{pmatrix} 6 \\ 9 \\ 1 \end{pmatrix} = \frac{\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} + \overrightarrow{OQ'}}{2}$$

$$\overrightarrow{OQ'} = \begin{pmatrix} 12 \\ 21 \\ 1 \end{pmatrix}$$

$$\overrightarrow{PQ'} = \begin{pmatrix} 12 \\ 21 \\ 1 \end{pmatrix} - \begin{pmatrix} 10 \\ 7 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 14 \\ -5 \end{pmatrix}$$

$$l_{PQ'} : \vec{r} = \begin{pmatrix} 10 \\ 7 \\ 6 \end{pmatrix} + s \begin{pmatrix} 2 \\ 14 \\ -5 \end{pmatrix}, s \in \mathbb{R}$$



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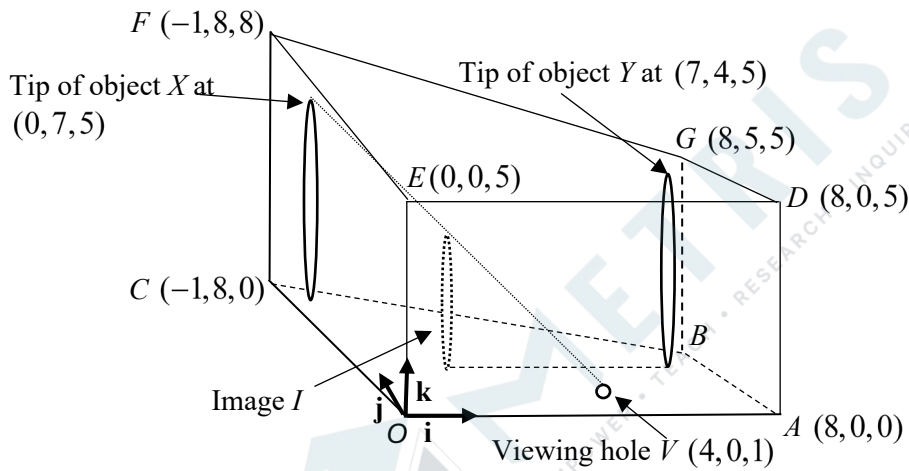
(a)

$$\overrightarrow{EF} = \begin{pmatrix} -1 \\ 8 \\ 8 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \\ 3 \end{pmatrix}, \quad \overrightarrow{EG} = \begin{pmatrix} 8 \\ 5 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ 0 \end{pmatrix}$$

$$\text{normal of plane } EFG = \begin{pmatrix} -1 \\ 8 \\ 3 \end{pmatrix} \times \begin{pmatrix} 8 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} -15 \\ 24 \\ -69 \end{pmatrix} = -3 \begin{pmatrix} 5 \\ -8 \\ 23 \end{pmatrix}$$

$$\mathbf{r} \cdot \begin{pmatrix} 5 \\ -8 \\ 23 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -8 \\ 23 \end{pmatrix} = 115 \Rightarrow 5x - 8y + 23z = 115 \text{ (shown)}$$

(b)



Let the upper tip of the Object X be point $X(0, 7, 5)$, and upper tip of Image I be point I .

$$\text{Line of sight has direction vector } \overrightarrow{XV} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \\ -4 \end{pmatrix},$$

$$\text{and given by the vector equation } \mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -7 \\ -4 \end{pmatrix}, \lambda \in \mathbb{R}.$$

Since the upper tip of Image I is also lying on this line and Image I is on the plane $y = 4$, we have $\lambda = -\frac{4}{7}$. Hence

$$\overrightarrow{OI} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} - \frac{4}{7} \begin{pmatrix} 4 \\ -7 \\ -4 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 12 \\ 28 \\ 23 \end{pmatrix}.$$

The ratio is $\frac{23}{7} : 5 \Rightarrow 23 : 35$ because image I has vertical height $\frac{23}{7}$ units and vertical height of object Y is 5 units.

(c)

Line BC is given by the vector equation $\vec{r} = \begin{pmatrix} -1 \\ 8 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \mu \in \mathbb{R}$.

S is the foot of perpendicular from the upper tip of Object $Y(7, 4, 5)$ to line BC and therefore

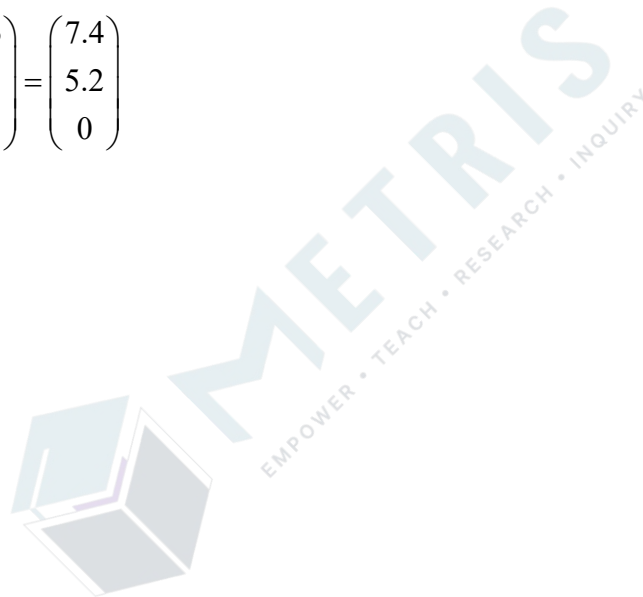
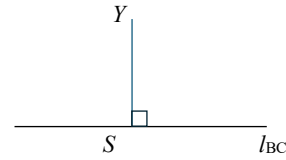
$$\vec{OS} = \begin{pmatrix} -1 \\ 8 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \text{ for some } \mu \in \mathbb{R}.$$

Since $\vec{SY}$ is perpendicular to the line BC ,

$$\left[\begin{pmatrix} -1 \\ 8 \\ 0 \end{pmatrix} - \begin{pmatrix} 7 \\ 4 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = 0$$
$$(24 + 4) + \mu(9 + 1) = 0$$

$$\mu = -2.8$$

$$\vec{OS} = \begin{pmatrix} -1 \\ 8 \\ 0 \end{pmatrix} - 2.8 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7.4 \\ 5.2 \\ 0 \end{pmatrix}$$



(i)

$$\sin \theta = \frac{\left| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \right|}{\sqrt{1}\sqrt{4^2 + 3^2 + 1^2}}$$

$$\sin \theta = \frac{1}{\sqrt{26}}$$

$$\theta = \sin^{-1} \frac{1}{\sqrt{26}}$$

$$\theta = 11.3^\circ$$

- (ii) Let the point on the flight path closest to the peak be N .
Let the peak be represented by point M .

$$\overrightarrow{OM} = \begin{pmatrix} 18000 \\ 4500 \\ 2000 \end{pmatrix}$$

Since N lies on the flight path,

$$\overrightarrow{ON} = \begin{pmatrix} 2000 \\ 500 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \text{ for some } \lambda.$$

$$\overrightarrow{NM} = \begin{pmatrix} 16000 - 4\lambda \\ 4000 - 3\lambda \\ 2000 - \lambda \end{pmatrix}$$

$$\overrightarrow{NM} \cdot \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} = 0$$

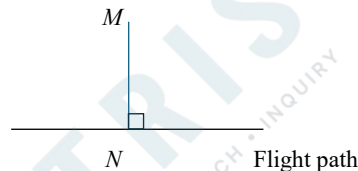
$$\begin{pmatrix} 16000 - 4\lambda \\ 4000 - 3\lambda \\ 2000 - \lambda \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} = 0$$

$$78000 = 26\lambda \Rightarrow \lambda = 3000$$

$$\therefore \overrightarrow{ON} = \begin{pmatrix} 2000 \\ 500 \\ 0 \end{pmatrix} + 3000 \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 14000 \\ 9500 \\ 3000 \end{pmatrix}$$

$$N(14000, 9500, 3000).$$

$$|\overrightarrow{NM}| = \left| \begin{pmatrix} 4000 \\ -5000 \\ -1000 \end{pmatrix} \right| = \sqrt{4000^2 + 5000^2 + 1000^2} = 6481 \text{ m}$$



(iii)

$$l_A : \vec{r} = \begin{pmatrix} 2000 \\ 500 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$$

$$l_B : \vec{r} = \begin{pmatrix} 1400 \\ 2900 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 2 \\ h \end{pmatrix}$$

For the flight paths to cross each other, there should be solutions to the set of linear equations:

$$\begin{cases} 2000 + 4\lambda = 1400 + 4\mu \\ 500 + 3\lambda = 2900 + 2\mu \\ \lambda = h\mu \end{cases} \Rightarrow \begin{cases} 4h\mu - 4\mu = -600 \\ 3h\mu - 2\mu = 2400 \\ h\mu - \mu = -150 \end{cases}$$

Solving simultaneously, $h\mu = 2700, \mu = 2850$

$$h = \frac{2700}{2850} = \frac{18}{19} = 0.947$$

(iv)

$$l_A : \vec{r} = \begin{pmatrix} 2000 \\ 500 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 1400 \\ 2900 \\ 0 \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} -600 \\ 2400 \\ 0 \end{pmatrix} = 600 \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$$

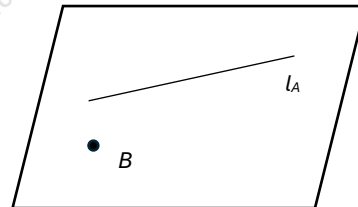
$$\overrightarrow{AB} \times \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} \times \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -19 \end{pmatrix}$$

$$\vec{r} \cdot \begin{pmatrix} 4 \\ 1 \\ -19 \end{pmatrix} = \begin{pmatrix} 2000 \\ 500 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ -19 \end{pmatrix}$$

$$\vec{r} \cdot \begin{pmatrix} 4 \\ 1 \\ -19 \end{pmatrix} = 8500$$

$$\therefore 4x + y - 19z = 8500$$

$$l_B : \vec{r} = \begin{pmatrix} 1400 \\ 2900 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$



$$\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ -19 \end{pmatrix} = -1 \neq 0$$

This implies that the direction vector of the flight path of Aircraft Bravo is not perpendicular to the normal of the found plane, and hence does not lie in the plane containing the flight path of Aircraft Alpha.

(As such, the flight path of Aircraft Bravo will not cross the flight path of Aircraft Alpha.)



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- (a) Let A and B be the take-off and anchor point respectively.

$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$= \begin{pmatrix} 25 + 2\mu \\ 20 + \mu \\ 20 + 2\mu \end{pmatrix} - \begin{pmatrix} 50 \\ 50 \\ 100 \end{pmatrix} = \begin{pmatrix} -25 + 2\mu \\ -30 + \mu \\ -80 + 2\mu \end{pmatrix}$$

$$|\vec{AB}| = 100$$

$$\sqrt{(-25 + 2\mu)^2 + (-30 + \mu)^2 + (-80 + 2\mu)^2} = 100$$

$$9\mu^2 - 480\mu - 2075 = 0$$

$$\mu = -4.02 \text{ or } 57.35 \text{ (rejected since the z-component will be 120.74)}$$

$$\vec{OB} = \begin{pmatrix} 25 + 2(-4.02) \\ 20 - 4.02 \\ 20 + 2(-4.02) \end{pmatrix} = \begin{pmatrix} 16.96 \\ 15.98 \\ 11.96 \end{pmatrix}$$

The zipline will end at the point (16.96, 15.98, 11.96).

- (b) **Method ①:**

Let θ denote the angle the zipline makes with the plane

$$\theta = \sin^{-1} \frac{\left| \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|}{\sqrt{10}\sqrt{1}}$$

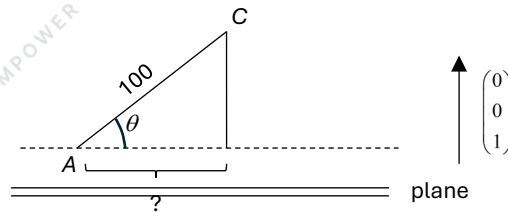
$$= \sin^{-1} \left| \frac{1}{\sqrt{10}} \right|$$

$$= 0.32175 \text{ rad}$$

Length of projection

$$= 100 \cos 0.32175$$

$$= 94.9$$



Method ②:

Let C be the new anchor point.

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$

$$\begin{aligned} &= \begin{pmatrix} 50+3\beta \\ 50 \\ 100+\beta \end{pmatrix} - \begin{pmatrix} 50 \\ 50 \\ 100 \end{pmatrix} \\ &= \begin{pmatrix} 3\beta \\ 0 \\ \beta \end{pmatrix} \end{aligned}$$

$$\left| \begin{pmatrix} 3\beta \\ 0 \\ \beta \end{pmatrix} \right| = 100$$

$$\beta\sqrt{10} = 100$$

$$\beta = \frac{100}{\sqrt{10}}$$

Length of shadow

$$= \frac{\left| \frac{100}{\sqrt{10}} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|}{\sqrt{1^2}}$$

$$= \frac{100}{\sqrt{10}} \left| \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} \right|$$

$$= \frac{100\sqrt{9}}{\sqrt{10}}$$

$$= \frac{300}{\sqrt{10}}$$

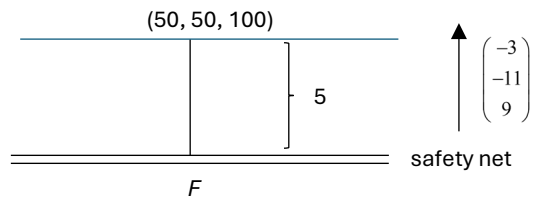
$$= 94.9$$



(c)

$$l_F : \vec{r} = \begin{pmatrix} 50 \\ 50 \\ 100 \end{pmatrix} + a \begin{pmatrix} -3 \\ -11 \\ 9 \end{pmatrix}, a \in \mathbb{R}$$

$$\overrightarrow{OF} = \begin{pmatrix} 50-3a \\ 50-11a \\ 100+9a \end{pmatrix} \text{ for some } a$$



$$\begin{pmatrix} 50 \\ 50 \\ 100 \end{pmatrix} - \begin{pmatrix} 50-3a \\ 50-11a \\ 100+9a \end{pmatrix} = \begin{pmatrix} 3a \\ 11a \\ -9a \end{pmatrix}$$

$$\left\| \begin{pmatrix} 3a \\ 11a \\ -9a \end{pmatrix} \right\| = 5$$

$$\sqrt{9a^2 + 121a^2 + 81a^2} = 5$$

$$211a^2 = 25$$

$$a = \frac{5}{\sqrt{211}} \text{ (rejected) or } -\frac{5}{\sqrt{211}}$$

$$\overrightarrow{OF} = \begin{pmatrix} 50-3\left(-\frac{5}{\sqrt{211}}\right) \\ 50-11\left(-\frac{5}{\sqrt{211}}\right) \\ 100+9\left(-\frac{5}{\sqrt{211}}\right) \end{pmatrix} = \begin{pmatrix} 50+\frac{15}{\sqrt{211}} \\ 50+\frac{55}{\sqrt{211}} \\ 100-\frac{45}{\sqrt{211}} \end{pmatrix}$$

$$\vec{r} \cdot \begin{pmatrix} -3 \\ -11 \\ 9 \end{pmatrix} = \begin{pmatrix} 50+\frac{15}{\sqrt{211}} \\ 50+\frac{55}{\sqrt{211}} \\ 100-\frac{45}{\sqrt{211}} \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -11 \\ 9 \end{pmatrix}$$

$$\vec{r} \cdot \begin{pmatrix} -3 \\ -11 \\ 9 \end{pmatrix} = -150 - \frac{45}{\sqrt{211}} - 550 - \frac{605}{\sqrt{211}} + 900 - \frac{405}{\sqrt{211}}$$

$$\vec{r} \cdot \begin{pmatrix} -3 \\ -11 \\ 9 \end{pmatrix} = 200 - \frac{1055}{\sqrt{211}}$$

$$\vec{r} \cdot \begin{pmatrix} -3 \\ -11 \\ 9 \end{pmatrix} = 200 - 5\sqrt{211}$$

$$-3x - 11y + 9z = 200 - 5\sqrt{211}$$

(d) $a \in \mathbb{R}, a \neq 0$ $\lambda > 0$ and $c < 0$ or $\lambda < 0$ and $c > 0$

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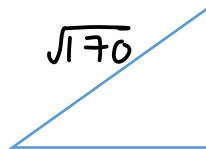
(a)

$$\vec{AB} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

$$\cos \theta_1 = \frac{\left| \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \right|}{\sqrt{(-1)^2 + 4^2} \sqrt{3^2 + (-1)^2}}$$

$$\cos \theta_1 = \frac{7}{\sqrt{17}\sqrt{10}} = \frac{7}{\sqrt{170}}$$

$$\sin \theta_1 = \sqrt{\frac{121}{170}}$$



$$\sqrt{121}$$

$$k = \frac{\sin \theta_1}{\sin \theta_2} = \frac{\sqrt{\frac{121}{170}}}{\frac{1}{\sqrt{2}}} = \sqrt{\frac{121}{85}}$$

(b)

$$\cos \frac{\pi}{4} = \frac{\left| \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} t \\ 0 \\ -1 \end{pmatrix} \right|}{\sqrt{(-1)^2 + 4^2} \sqrt{t^2 + (-1)^2}}$$

$$\frac{1}{\sqrt{2}} = \frac{|-t-4|}{\sqrt{17}\sqrt{t^2+1}}$$

$$\frac{1}{\sqrt{2}} = \frac{|t+4|}{\sqrt{17}\sqrt{t^2+1}}$$

$$\frac{1}{\sqrt{2}} = \frac{t+4}{\sqrt{17}\sqrt{t^2+1}} \text{ as } t > 0$$

$$\frac{1}{2} = \frac{(t+4)^2}{17(t^2+1)}$$

$$17t^2 + 17 = 2t^2 + 16t + 32$$

$$15t^2 - 16t - 15 = 0$$

$$(3t-5)(5t+3) = 0$$

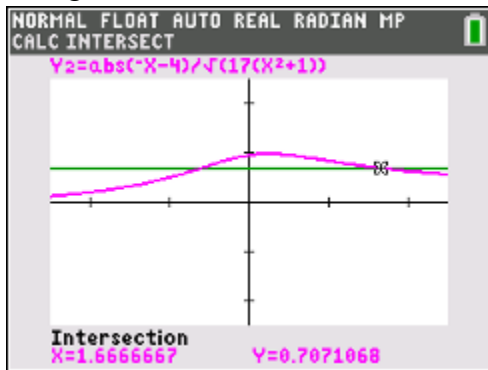
$$\Rightarrow t = \frac{5}{3} \text{ or } -\frac{3}{5} (\text{rejected})$$

$$\text{Equation of the line } BC \text{ is } \underline{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} \frac{5}{3} \\ 0 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

Method ②:

$$\cos \frac{\pi}{4} = \frac{\begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} t \\ 0 \\ -1 \end{pmatrix}}{\sqrt{(-1)^2 + 4^2} \sqrt{t^2 + (-1)^2}} = \frac{|-t-4|}{\sqrt{17(t^2+1)}}$$

Using G.C.,



Since $t > 0 \Rightarrow t = \frac{5}{3}$

Equation of the line BC is $\vec{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} \frac{5}{3} \\ 0 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}$

(c)

Equation of the bottom surface is $\vec{r} \cdot \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$

i.e. $\vec{r} \cdot \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = -1$

(d)

$$h = \text{Distance of } E \text{ from top surface} = \frac{\left| \vec{BE} \cdot \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} \right|}{\sqrt{(-1)^2 + 4^2}}$$

$$\begin{aligned} &= \frac{\left| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right| \cdot \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}}{\sqrt{17}} \\ &= \frac{2\sqrt{17}}{17} \end{aligned}$$

(e) Since C lies on the line BC,

$$\overrightarrow{OC} = \begin{pmatrix} 3 + \frac{5}{3}\lambda \\ 0 \\ 1 - \lambda \end{pmatrix} \text{ for some } \lambda$$

Since C is the intersection between line BC and bottom surface,

$$\begin{pmatrix} 3 + \frac{5}{3}\lambda \\ 0 \\ 1 - \lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = -1$$

$$1 - \frac{17}{3}\lambda = -1$$

$$\lambda = \frac{6}{17}$$

$$\therefore \overrightarrow{OC} = \begin{pmatrix} \frac{61}{17} \\ 0 \\ \frac{11}{17} \end{pmatrix}$$

Equation of line CD is $\vec{r} = \overrightarrow{OC} + \mu \overrightarrow{AB}$, $\mu \in \mathbb{R}$

$$\text{i.e. } \vec{r} = \begin{pmatrix} \frac{61}{17} \\ 0 \\ \frac{11}{17} \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \quad \mu \in \mathbb{R}$$

Since point D is on the line CD $\Rightarrow \overrightarrow{OD} = \begin{pmatrix} \frac{61}{17} + 3\mu \\ 0 \\ \frac{11}{17} - \mu \end{pmatrix}$

At D, $z = 0$,

$$\frac{11}{17} - \mu = 0 \Rightarrow \mu = \frac{11}{17}$$

$$\overrightarrow{OD} = \begin{pmatrix} \frac{61}{17} \\ 0 \\ \frac{11}{17} \end{pmatrix} + \frac{11}{17} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{94}{17} \\ 0 \\ 0 \end{pmatrix}$$