

1

$$\int_{-1}^{a-1} x\sqrt{x+1} \, dx = \int_0^{\sqrt{a}} x\sqrt{x+1} \frac{dx}{du} du$$

$$= \int_0^{\sqrt{a}} (u^2 - 1)(u)(2u) \, du$$

$$= 2 \int_0^{\sqrt{a}} u^4 - u^2 \, du$$

$$= 2 \left[\frac{1}{5} u^5 - \frac{1}{3} u^3 \right]_0^{\sqrt{a}}$$

$$= 2 \left[\left(\frac{1}{5} (\sqrt{a})^5 - \frac{1}{3} (\sqrt{a})^3 \right) - \left(\frac{1}{5} (0)^5 - \frac{1}{3} (0)^3 \right) \right]$$

$$= \frac{2}{5} a^{\frac{5}{2}} - \frac{2}{3} a^{\frac{3}{2}}$$

$$u^2 = x + 1$$

$$x = u^2 - 1$$

$$\frac{dx}{du} = 2u$$

$$\text{When } x = a - 1, \quad u = \sqrt{a}$$

$$\text{When } x = -1, \quad u = 0$$



2

(a)

$$\begin{aligned} & \int \frac{2x^3}{1+x^4} dx \\ &= \frac{1}{2} \int \frac{4x^3}{1+x^4} dx \\ &= \frac{1}{2} \ln|1+x^4| + c \\ &= \frac{1}{2} \ln(1+x^4) + c \quad \text{since } 1+x^4 > 0 \end{aligned}$$

(b)

$$\begin{aligned} \int_0^1 \frac{x^3}{(x^2+1)^3} dx &= \int_0^1 \frac{x^3}{(x^2+1)^3} \frac{dx}{d\theta} d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{\tan^3 \theta}{(\tan^2 \theta + 1)^3} \sec^2 \theta d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{\tan^3 \theta}{(\sec^2 \theta)^3} \sec^2 \theta d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{\tan^3 \theta}{\sec^4 \theta} d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{\sin^3 \theta}{\cos^3 \theta} \cdot \cos^4 \theta d\theta \\ &= \int_0^{\frac{\pi}{4}} \cos \theta \sin^3 \theta d\theta \quad (\text{Shown}) \end{aligned}$$

$$x = \tan \theta$$

$$\frac{dx}{d\theta} = \sec^2 \theta$$

$$\text{When } x = 1, \theta = \frac{\pi}{4}$$

$$\text{When } x = 0, \theta = 0$$

Hence,

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \cos \theta \sin^3 \theta d\theta &= \left[\frac{\sin^4 \theta}{4} \right]_0^{\frac{\pi}{4}} \\ &= \frac{\left(\sin \frac{\pi}{4} \right)^4}{4} - \frac{(\sin 0)^4}{4} \\ &= \frac{\left(\frac{1}{\sqrt{2}} \right)^4}{4} \\ &= \frac{1}{16} \end{aligned}$$

3

(a)(i)

$$\begin{aligned}\int x^2 \sqrt{x^3+1} \, dx &= \frac{1}{3} \int 3x^2 (x^3+1)^{\frac{1}{2}} \, dx \\ &= \frac{1}{3} \left(\frac{(x^3+1)^{\frac{3}{2}}}{\frac{3}{2}} \right) + C \\ &= \frac{2}{9} (x^3+1)^{\frac{3}{2}} + C\end{aligned}$$

(a)(ii)

$$\begin{aligned}\int_{-1}^0 x^5 \sqrt{x^3+1} \, dx &= \int_{-1}^0 x^3 \cdot x^2 \sqrt{x^3+1} \, dx \\ &= \left[\frac{2}{9} x^3 (x^3+1)^{\frac{3}{2}} \right]_{-1}^0 - \int_{-1}^0 (3x^2) \cdot \frac{2}{9} (x^3+1)^{\frac{3}{2}} \, dx \\ &= \left[0 - \frac{2}{9} (-1)^3 ((-1)^3+1)^{\frac{3}{2}} \right] - \frac{2}{9} \int_{-1}^0 (3x^2) \cdot (x^3+1)^{\frac{3}{2}} \, dx \\ &= 0 - \frac{2}{9} \left[\frac{(x^3+1)^{\frac{5}{2}}}{\frac{5}{2}} \right]_{-1}^0 \\ &= -\frac{4}{45} \left[(x^3+1)^{\frac{5}{2}} \right]_{-1}^0 \\ &= -\frac{4}{45} (1-0) \\ &= -\frac{4}{45}\end{aligned}$$

$$\begin{array}{l} u = x^3 \quad \frac{dv}{dx} = x^2 \sqrt{x^3+1} \\ \frac{du}{dx} = 3x^2 \quad \leftarrow v = \frac{2}{9} (x^3+1)^{\frac{3}{2}} \end{array}$$

(b)

$$\begin{aligned}\int e^{2x} \sqrt{e^x+1} \, dx &= \int e^{2x} \sqrt{e^x+1} \frac{dx}{du} du \\ &= \int e^{2x} \sqrt{e^x+1} \frac{1}{e^x} du \\ &= \int e^x \sqrt{e^x+1} \, du \\ &= \int (u-1) u^{\frac{1}{2}} \, du \\ &= \int u^{\frac{3}{2}} - u^{\frac{1}{2}} \, du \\ &= \frac{2}{5} u^{\frac{5}{2}} - \frac{2}{3} u^{\frac{3}{2}} + C \\ &= \frac{2}{5} (\sqrt{1+e^x})^5 - \frac{2}{3} (\sqrt{1+e^x})^3 + C\end{aligned}$$

$$\begin{array}{l} u = 1 + e^x \Rightarrow e^x = u - 1 \\ \frac{du}{dx} = e^x \\ \frac{dx}{du} = \frac{1}{e^x} \end{array}$$

4

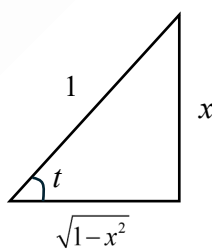
(a) $\frac{d}{dx}(e^{x^2+2x}) = (2x+2)e^{x^2+2x}$

$$\begin{aligned}\int_0^1 (x+1)e^{x^2+2x} dx &= \frac{1}{2} \int_0^1 2(x+1)e^{x^2+2x} dx \\ &= \frac{1}{2} [e^{x^2+2x}]_0^1 \\ &= \frac{1}{2} [e^3 - e^0] \\ &= \frac{1}{2} (e^3 - 1)\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{1}{(1-x^2)^{\frac{3}{2}}} dx &= \int \frac{1}{(1-x^2)^{\frac{3}{2}}} \frac{dx}{dt} dt \\ &= \int \frac{1}{(1-\sin^2 t)^{\frac{3}{2}}} \cos t dt \\ &= \int \frac{1}{(\cos^2 t)^{\frac{3}{2}}} \cos t dt \\ &= \int \frac{1}{(\cos^3 t)} \cos t dt \\ &= \int \frac{1}{(\cos^2 t)} dt \\ &= \int \sec^2 t dt \\ &= \tan t + C \\ &= \frac{x}{\sqrt{1-x^2}} + C\end{aligned}$$

$$\left| \begin{array}{l} x = \sin t \\ \frac{dx}{dt} = \cos t \end{array} \right.$$



5

(a)

$$\begin{aligned}
 \int \frac{x^5}{\sqrt{1-x^3}} dx &= \int \frac{x^5}{\sqrt{1-x^3}} \frac{dx}{du} du \\
 &= \int \frac{x^5}{\sqrt{1-x^3}} \left(-\frac{1}{3x^2} \right) du \\
 &= -\frac{1}{3} \int \frac{x^3}{\sqrt{1-x^3}} du \\
 &= -\frac{1}{3} \int \frac{1-u}{\sqrt{u}} du \\
 &= -\frac{1}{3} \int \frac{1}{\sqrt{u}} - \frac{u}{\sqrt{u}} du \\
 &= -\frac{1}{3} \int u^{-\frac{1}{2}} - u^{\frac{1}{2}} du \\
 &= -\frac{1}{3} \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right] + C \\
 &= -\frac{2}{3} u^{\frac{1}{2}} + \frac{2}{9} u^{\frac{3}{2}} + C \\
 &= -\frac{2}{3} (1-x^3)^{\frac{1}{2}} + \frac{2}{9} (1-x^3)^{\frac{3}{2}} + C
 \end{aligned}$$

$$\begin{aligned}
 u &= 1-x^3 \Rightarrow x^3 = 1-u \\
 \frac{du}{dx} &= -3x^2 \\
 \frac{dx}{du} &= -\frac{1}{3x^2}
 \end{aligned}$$

(b)

$$\begin{aligned}
 &\int \frac{x^8}{\sqrt{1-x^3}} dx \\
 &= \int x^3 \left(\frac{x^5}{\sqrt{1-x^3}} \right) dx \\
 &= x^3 \left[-\frac{2}{3} (1-x^3)^{\frac{1}{2}} + \frac{2}{9} (1-x^3)^{\frac{3}{2}} \right] - \int 3x^2 \left(-\frac{2}{3} (1-x^3)^{\frac{1}{2}} + \frac{2}{9} (1-x^3)^{\frac{3}{2}} \right) dx \\
 &= -\frac{2}{3} x^3 (1-x^3)^{\frac{1}{2}} + \frac{2}{9} x^3 (1-x^3)^{\frac{3}{2}} - \frac{2}{3} \int (-3x^2) (1-x^3)^{\frac{1}{2}} dx + \frac{2}{9} \int (-3x^2) (1-x^3)^{\frac{3}{2}} dx \\
 &= -\frac{2}{3} x^3 (1-x^3)^{\frac{1}{2}} + \frac{2}{9} x^3 (1-x^3)^{\frac{3}{2}} - \frac{2}{3} \left[\frac{(1-x^3)^{\frac{3}{2}}}{\frac{3}{2}} \right] + \frac{2}{9} \left[\frac{(1-x^3)^{\frac{5}{2}}}{\frac{5}{2}} \right] + C \\
 &= -\frac{2}{3} x^3 (1-x^3)^{\frac{1}{2}} + \frac{2}{9} x^3 (1-x^3)^{\frac{3}{2}} - \frac{4}{9} (1-x^3)^{\frac{3}{2}} + \frac{4}{45} (1-x^3)^{\frac{5}{2}} + C
 \end{aligned}$$

$$\begin{aligned}
 u &= x^3 \quad \frac{dv}{dx} = \frac{x^5}{\sqrt{1-x^3}} \\
 \frac{du}{dx} &= 3x^2 \quad \leftarrow v = -\frac{2}{3} (1-x^3)^{\frac{1}{2}} + \frac{2}{9} (1-x^3)^{\frac{3}{2}}
 \end{aligned}$$

6

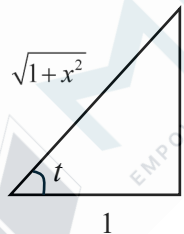
(a)

$$\begin{aligned}
 \int \frac{1}{x^2 \sqrt{x^2 + 1}} dx &= \int \frac{1}{x^2 \sqrt{x^2 + 1}} \frac{dx}{dt} dt \\
 &= \int \frac{1}{\tan^2 t \sqrt{\tan^2 t + 1}} (\sec^2 t) dt \\
 &= \int \frac{1}{\tan^2 t (\sec t)} (\sec^2 t) dt \\
 &= \int \frac{\sec t}{\tan^2 t} dt \\
 &= \int \frac{1}{\cos t} \cdot \frac{\cos^2 t}{\sin^2 t} dt \\
 &= \int \frac{\cos t}{\sin^2 t} dt \\
 &= \int (\cos t) (\sin t)^{-2} dt \\
 &= \frac{(\sin t)^{-2+1}}{-1} + C \\
 &= -\frac{1}{\sin t} + C \\
 &= -\frac{1}{\frac{x}{\sqrt{1+x^2}}} + C \\
 &= -\frac{\sqrt{1+x^2}}{x} + C
 \end{aligned}$$

$$\begin{aligned}
 x = \tan t &\Rightarrow x^2 = 1 - u \\
 \frac{dx}{dt} &= \sec^2 t
 \end{aligned}$$

Alternatively,

$$\begin{aligned}
 \int \frac{\cos t}{\sin^2 t} dt &= \int \cos \operatorname{csc} t \cot t dt \\
 &= -\cos \operatorname{csc} t + C \\
 &= -\frac{1}{\sin t} + C
 \end{aligned}$$



(b)(i) $\frac{d}{dx}(\cos x^3) = -3x^2 \sin x^3$

(b)(ii)

$$\begin{aligned}
 &\int x^5 \sin x^3 dx \\
 &= -\frac{1}{3} \int (x^3) (-3x^2 \sin x^3) dx \\
 &= -\frac{1}{3} \left[x^3 \cos x^3 - \int 3x^2 \cos x^3 dx \right] \\
 &= -\frac{x^3}{3} \cos x^3 + \frac{1}{3} \int 3x^2 \cos x^3 dx \\
 &= -\frac{x^3}{3} \cos x^3 + \frac{1}{3} \sin x^3 + C
 \end{aligned}$$

7

(a)

$$\begin{aligned}
 \frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
 &= \frac{\cos^2 x - \sin x(-\sin x)}{\cos^2 x} \\
 &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
 &= \frac{1}{\cos^2 x} \\
 &= \sec^2 x \quad (\text{shown})
 \end{aligned}$$

(b)

$$\begin{aligned}
 \sin 2x \cot x &= 2 \sin x \cos x \left(\frac{\cos x}{\sin x}\right) \\
 &= 2 \cos^2 x \quad (\text{shown})
 \end{aligned}$$

(c)

$$\begin{aligned}
 \int_{\frac{\pi}{30}}^{\frac{\pi}{15}} \operatorname{cosec}(10x) \tan(5x) \, dx &= \int_{\frac{\pi}{30}}^{\frac{\pi}{15}} \frac{1}{\sin(10x) \cot(5x)} \, dx \\
 &= \int_{\frac{\pi}{30}}^{\frac{\pi}{15}} \frac{1}{2 \cos^2 5x} \, dx \\
 &= \frac{1}{2} \int_{\frac{\pi}{30}}^{\frac{\pi}{15}} \sec^2 5x \, dx \\
 &= \frac{1}{2} \cdot \frac{1}{5} \int_{\frac{\pi}{30}}^{\frac{\pi}{15}} 5 \sec^2 5x \, dx \\
 &= \frac{1}{10} [\tan 5x]_{\frac{\pi}{30}}^{\frac{\pi}{15}} \\
 &= \frac{1}{10} \left[\tan \frac{\pi}{3} - \tan \frac{\pi}{6} \right] \\
 &= \frac{1}{10} \left[\sqrt{3} - \frac{1}{\sqrt{3}} \right] \\
 &= \frac{1}{10} \left[\sqrt{3} - \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \right] \\
 &= \frac{1}{10} \left[\frac{2\sqrt{3}}{3} \right] \\
 &= \frac{\sqrt{3}}{15}
 \end{aligned}$$

8

(a)

$$\begin{aligned}
 \int \cot^2 3x \, dx &= \int \operatorname{cosec}^2 3x - 1 \, dx \\
 &= \frac{1}{3} \int 3 \operatorname{cosec}^2 3x \, dx - \int 1 \, dx \\
 &= -\frac{\cot 3x}{3} - x + C
 \end{aligned}$$

(b)

$$\begin{aligned}
 \int_2^4 \frac{x^2 - 4x + 3}{x^2 - 4x + 8} \, dx &= \int_2^4 \frac{x^2 - 4x + 8 - 5}{x^2 - 4x + 8} \, dx \\
 &= \int_2^4 \frac{x^2 - 4x + 8}{x^2 - 4x + 8} - \frac{5}{x^2 - 4x + 8} \, dx \\
 &= \int_2^4 1 - \frac{5}{(x-2)^2 + 4} \, dx \\
 &= \int_2^4 1 \, dx - 5 \int_2^4 \frac{1}{(x-2)^2 + 2^2} \, dx \\
 &= \left[x - \frac{5}{2} \tan^{-1} \left(\frac{x-2}{2} \right) \right]_2^4 \\
 &= \left[4 - \frac{5}{2} \tan^{-1}(1) \right] - \left[2 - \frac{5}{2} \tan^{-1}(0) \right] \\
 &= 4 - \frac{5}{2} \left(\frac{\pi}{4} \right) - 2 \\
 &= 2 - \frac{5\pi}{8}
 \end{aligned}$$

(c)

$$\begin{aligned}\int_{\frac{2\sqrt{3}}{3}}^2 \frac{4-x^2}{(4+x^2)^2} dx &= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{4-x^2}{(4+x^2)^2} \frac{dx}{d\theta} d\theta \\&= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{4-4\cot^2 \theta}{(4+4\cot^2 \theta)^2} (-2\operatorname{cosec}^2 \theta) d\theta \\&= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{4-4\cot^2 \theta}{(4\operatorname{cosec}^2 \theta)^2} (-2\operatorname{cosec}^2 \theta) d\theta \\&= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{4(1-\cot^2 \theta)}{16\operatorname{cosec}^4 \theta} (-2\operatorname{cosec}^2 \theta) d\theta \\&= -\frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1-\cot^2 \theta}{\operatorname{cosec}^2 \theta} d\theta \\&= -\frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1}{\operatorname{cosec}^2 \theta} - \frac{\cot^2 \theta}{\operatorname{cosec}^2 \theta} d\theta \\&= -\frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \sin^2 \theta - \cos^2 \theta d\theta \\&= \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \cos^2 \theta - \sin^2 \theta d\theta \\&= \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \cos 2\theta d\theta \\&= \frac{1}{2} \left[\frac{\sin 2\theta}{2} \right]_{\frac{\pi}{3}}^{\frac{\pi}{4}} \\&= \frac{1}{4} [\sin 2\theta]_{\frac{\pi}{3}}^{\frac{\pi}{4}} \\&= \frac{1}{4} \left[\sin \frac{\pi}{2} - \sin \frac{2\pi}{3} \right] \\&= \frac{1}{4} \left[1 - \frac{\sqrt{3}}{2} \right] \\&= \frac{1}{4} - \frac{\sqrt{3}}{8}\end{aligned}$$

$$x = 2 \cot \theta$$

$$\frac{dx}{d\theta} = -2 \operatorname{cosec}^2 \theta$$

$$\text{When } x = 2, \quad 2 \cot \theta = 2$$

$$\cot \theta = 1$$

$$\tan \theta = 1$$

$$\theta = \frac{\pi}{4}$$

$$\text{When } x = \frac{2\sqrt{3}}{3}, \quad 2 \cot \theta = \frac{2\sqrt{3}}{3}$$

$$\cot \theta = \frac{\sqrt{3}}{3}$$

$$\tan \theta = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$\theta = \frac{\pi}{3}$$

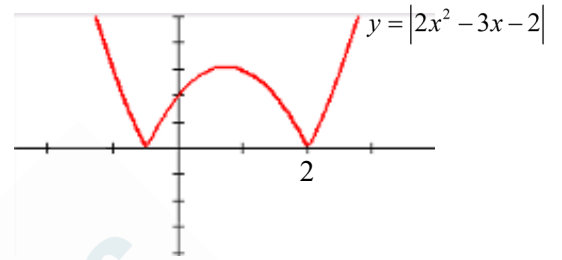
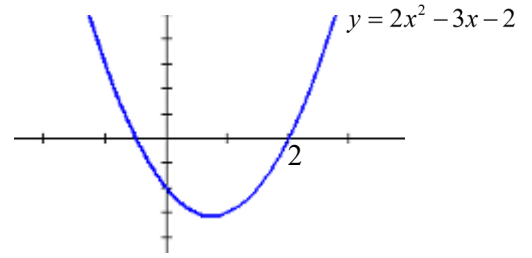
$$\begin{aligned}
\int \frac{\sin 2x}{\cos^2 x + \cos x + 2} dx &= \int \frac{\sin 2x}{\cos^2 x + \cos x + 2} \frac{dx}{du} du \\
&= \int \frac{2 \sin x \cos x}{\cos^2 x + \cos x + 2} \left(-\frac{1}{\sin x} \right) du \\
&= - \int \frac{2 \cos x}{\cos^2 x + \cos x + 2} du \\
&= - \int \frac{2u}{u^2 + u + 2} du \\
&= - \int \frac{2u + 1 - 1}{u^2 + u + 2} du \\
&= - \int \frac{2u + 1}{u^2 + u + 2} du - \int \frac{1}{u^2 + u + 2} du \\
&= - \int \frac{2u + 1}{u^2 + u + 2} du + \int \frac{1}{\left(u + \frac{1}{2}\right)^2 + \frac{7}{4}} du \\
&= - \ln |u^2 + u + 2| + \frac{1}{\sqrt{\frac{7}{4}}} \tan^{-1} \left(\frac{u + \frac{1}{2}}{\sqrt{\frac{7}{4}}} \right) + C \\
&= - \ln (u^2 + u + 2) + \frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{u + \frac{1}{2}}{\frac{\sqrt{7}}{2}} \right) + C \\
&= - \ln (u^2 + u + 2) + \frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{2u + 1}{\sqrt{7}} \right) + C \\
&= - \ln (\cos^2 x + \cos x + 2) + \frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{2 \cos x + 1}{\sqrt{7}} \right) + C
\end{aligned}$$

$$u = \cos x$$

$$\frac{du}{dx} = -\sin x$$

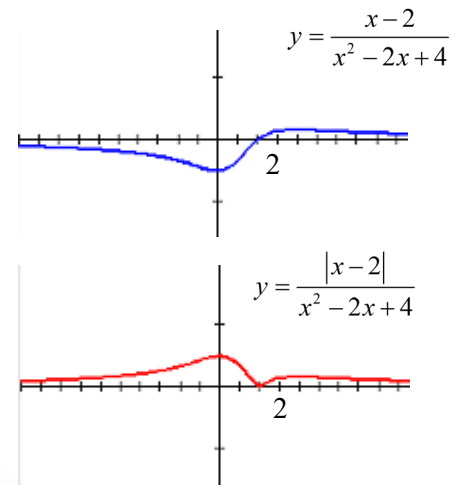
$$\frac{dx}{du} = -\frac{1}{\sin x}$$

$$\begin{aligned}
& \int_0^a |2x^2 - 3x - 2| \, dx \\
&= \int_0^2 -(2x^2 - 3x - 2) \, dx + \int_2^a (2x^2 - 3x - 2) \, dx \\
&= -\left[\frac{2}{3}x^3 - \frac{3}{2}x^2 - 2x\right]_0^2 + \left[\frac{2}{3}x^3 - \frac{3}{2}x^2 - 2x\right]_2^a \\
&= -\left[\frac{2}{3}(2)^3 - \frac{3}{2}(2)^2 - 2(2) - 0\right] \\
&\quad + \left[\left(\frac{2}{3}a^3 - \frac{3}{2}a^2 - 2a\right) - \left(\frac{2}{3}(2)^3 - \frac{3}{2}(2)^2 - 2(2)\right)\right] \\
&= -\left(-\frac{14}{3}\right) + \frac{2}{3}a^3 - \frac{3}{2}a^2 - 2a - \left(-\frac{14}{3}\right) \\
&= \frac{2}{3}a^3 - \frac{3}{2}a^2 - 2a + \frac{28}{3} \\
&= \frac{4}{6}a^3 - \frac{9}{6}a^2 - \frac{12}{6}a + \frac{56}{6} \\
&= \frac{1}{6}(4a^3 - 9a^2 - 12a + 56) \quad \text{where } k = 6
\end{aligned}$$



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$$\begin{aligned}
& \int_{-2}^2 \frac{|x-2|}{x^2-2x+4} dx \\
&= \int_{-2}^2 \frac{2-x}{x^2-2x+4} dx \\
&= \int_{-2}^2 \frac{-\frac{1}{2}(2x-2)+1}{x^2-2x+4} dx \\
&= -\frac{1}{2} \int_{-2}^2 \frac{2-x}{x^2-2x+4} dx + \int_{-2}^2 \frac{1}{(x-1)^2+3} dx \\
&= -\frac{1}{2} \left[\ln|x^2-2x+4| \right]_{-2}^2 + \left[\frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x-1}{\sqrt{3}} \right) \right]_{-2}^2 \\
&= -\frac{1}{2} [\ln 4 - \ln 12] + \left[\frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{1}{\sqrt{3}} \right) - \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{-2-1}{\sqrt{3}} \right) \right] \\
&= -\frac{1}{2} \ln \left(\frac{4}{12} \right) + \frac{1}{\sqrt{3}} \left[\tan^{-1} \left(\frac{1}{\sqrt{3}} \right) - \tan^{-1} (-\sqrt{3}) \right] \\
&= -\frac{1}{2} \ln \left(\frac{1}{3} \right) + \frac{1}{\sqrt{3}} \left[\frac{\pi}{6} - \left(-\frac{\pi}{3} \right) \right] \\
&= \frac{1}{2} \ln 3 + \frac{1}{\sqrt{3}} \left(\frac{\pi}{2} \right) \\
&= \frac{1}{2} \ln 3 + \frac{\pi}{2\sqrt{3}} \quad \text{where } p = \frac{1}{2}, q = \frac{1}{2\sqrt{3}} \text{ (shown)}
\end{aligned}$$



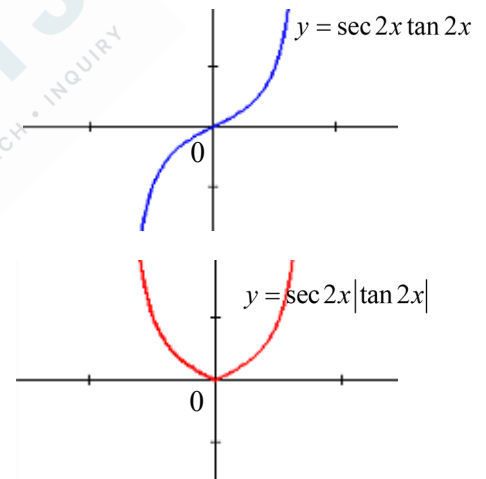
12

(a)

$$\begin{aligned}
 & \int \frac{3-2x}{\sqrt{8+2x-x^2}} dx \\
 &= \int \left(\frac{2-2x}{\sqrt{8+2x-x^2}} + \frac{1}{\sqrt{8+2x-x^2}} \right) dx \\
 &= \int (2-2x)(8+2x-x^2)^{-\frac{1}{2}} dx + \int \frac{1}{\sqrt{3^2-(x-1)^2}} dx \\
 &= \frac{(8+2x-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + \sin^{-1}\left(\frac{x-1}{3}\right) + C \\
 &= 2\sqrt{8+2x-x^2} + \sin^{-1}\left(\frac{x-1}{3}\right) + C
 \end{aligned}$$

(b)

$$\begin{aligned}
 & \int_{-\frac{\pi}{8}}^{\frac{\pi}{6}} \sec 2x |\tan 2x| dx \\
 &= \int_{-\frac{\pi}{8}}^0 -\sec 2x \tan 2x dx + \int_0^{\frac{\pi}{6}} \sec 2x \tan 2x dx \\
 &= -\frac{1}{2} \int_{-\frac{\pi}{8}}^0 2 \sec 2x \tan 2x dx + \frac{1}{2} \int_0^{\frac{\pi}{6}} 2 \sec 2x \tan 2x dx \\
 &= -\frac{1}{2} [\sec 2x]_{-\frac{\pi}{8}}^0 + \frac{1}{2} [\sec 2x]_0^{\frac{\pi}{6}} \\
 &= -\frac{1}{2} \left[\frac{1}{\cos 0} - \frac{1}{\cos\left(-\frac{\pi}{4}\right)} \right] + \frac{1}{2} \left[\frac{1}{\cos \frac{\pi}{3}} - \frac{1}{\cos(0)} \right] \\
 &= -\frac{1}{2} [1 - \sqrt{2}] + \frac{1}{2} [2 - 1] \\
 &= \frac{\sqrt{2}}{2}
 \end{aligned}$$



(c)

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \frac{1}{1+8\cos^2 x} dx &= \int_0^{\sqrt{3}} \frac{1}{1+8\cos^2 x} \frac{dx}{dt} dt \\&= \int_0^{\sqrt{3}} \frac{1}{1+\frac{8}{\sec^2 x}} \left(\frac{1}{\sec^2 x} \right) dt \\&= \int_0^{\sqrt{3}} \frac{1}{\sec^2 x + 8} dt \\&= \int_0^{\sqrt{3}} \frac{1}{\tan^2 x + 9} dt \\&= \int_0^{\sqrt{3}} \frac{1}{t^2 + 3^2} dt \\&= \left[\frac{1}{3} \tan^{-1} \frac{t}{3} \right]_0^{\sqrt{3}} \\&= \frac{\pi}{18}\end{aligned}$$

$$t = \tan x$$

$$\frac{dt}{dx} = \sec^2 x$$

$$\frac{dx}{dt} = \frac{1}{\sec^2 x}$$

$$\begin{aligned}\text{When } x = 0, \quad t &= \tan 0 \\t &= 0\end{aligned}$$

$$\begin{aligned}\text{When } x = \frac{\pi}{3}, \quad t &= \tan \frac{\pi}{3} \\t &= \sqrt{3}\end{aligned}$$



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$$(a)(i) \quad \frac{d}{dx} [\sin^{-1}(e^{2x})] = \frac{1}{\sqrt{1-(e^{2x})^2}} (e^{2x})(2) = \frac{2e^{2x}}{\sqrt{1-e^{4x}}}.$$

(a)(ii)

$$\begin{aligned} \int \frac{2e^{2x} + e^{4x}}{\sqrt{1-e^{4x}}} dx &= \int \frac{2e^{2x}}{\sqrt{1-e^{4x}}} dx + \int \frac{e^{4x}}{\sqrt{1-e^{4x}}} dx \\ &= \underbrace{\sin^{-1}(e^{2x})}_{\text{[From (a)(i)]}} - \frac{1}{4} \int -4e^{4x} (1-e^{4x})^{-\frac{1}{2}} dx \\ &= \sin^{-1}(e^{2x}) - \frac{1}{4} \left[\frac{(1-e^{4x})^{\frac{1}{2}}}{\frac{1}{2}} \right] + c \\ &= \sin^{-1}(e^{2x}) - \frac{1}{4} [2\sqrt{1-e^{4x}}] + c \\ &= \sin^{-1}(e^{2x}) - \frac{1}{2} \sqrt{1-e^{4x}} + c \end{aligned}$$

(b)(i)

$$\begin{aligned} \sin^{n-2} \theta \cos^2 \theta &= \sin^{n-2} \theta (1 - \sin^2 \theta) \\ &= \sin^{n-2} \theta - \sin^n \theta \quad (\text{shown}) \end{aligned}$$

(b)(ii)

$$\int_0^{\frac{\pi}{2}} \sin^n \theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{n-1} \theta \sin \theta d\theta$$

$$= \left[(\sin^{n-1} \theta)(-\cos \theta) \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos \theta) [(n-1) \sin^{n-2} \theta \cos \theta] d\theta$$

$$= [0 - 0] + (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta \cos^2 \theta d\theta$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta - \sin^n \theta d\theta \quad [\text{from (i)}]$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta d\theta - (n-1) \int_0^{\frac{\pi}{2}} \sin^n \theta d\theta$$

$$n \int_0^{\frac{\pi}{2}} \sin^n \theta d\theta = (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta d\theta$$

$$\int_0^{\frac{\pi}{2}} \sin^n \theta d\theta = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta d\theta \quad (\text{shown})$$

$$\begin{array}{lcl} u = \sin^{n-1} \theta & \xrightarrow{\quad} & \frac{dv}{d\theta} = \sin \theta \\ \frac{du}{d\theta} = (n-1) \sin^{n-2} \theta \cos \theta & \xleftarrow{\quad} & v = -\cos \theta \end{array}$$

(b)(iii) Substitute $n = 3$,

$$\int_0^{\frac{\pi}{2}} \sin^3 \theta \, d\theta = \left(\frac{3-1}{3} \right) \int_0^{\frac{\pi}{2}} \sin \theta \, d\theta$$

$$= \frac{2}{3} [-\cos \theta]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{3} [0 - (-1)]$$

$$= \frac{2}{3}$$



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(a) Method 1: Using substitution

$$\begin{aligned}
 \int_1^2 \frac{e^x}{(1-e^x)^3} dx &= \int_{1-e}^{1-e^2} \frac{e^x}{(1-e^x)^3} \frac{dx}{du} du \\
 &= \int_{1-e}^{1-e^2} \frac{e^x}{(1-e^x)^3} \left(-\frac{1}{e^x}\right) du \\
 &= \int_{1-e}^{1-e^2} \frac{-1}{u^3} du \\
 &= \left[\frac{1}{2} u^{-2} \right]_{1-e}^{1-e^2} \\
 &= \frac{1}{2} \left[(1-e^2)^{-2} - (1-e)^{-2} \right]
 \end{aligned}$$

$$u = 1 - e^x$$

$$\frac{du}{dx} = -e^x$$

$$\frac{dx}{du} = -\frac{1}{e^x}$$

$$\text{When } x = 1, \quad u = 1 - e^1$$

$$\text{When } x = 2, \quad u = 1 - e^2$$

Method 2: Otherwise

$$\begin{aligned}
 \int_1^2 \frac{e^x}{(1-e^x)^3} dx &= -\int_1^2 (-e^x)(1-e^x)^{-3} dx \\
 &= -\left[\frac{(1-e^x)^{-2}}{-2} \right]_1^2 \\
 &= \frac{1}{2} \left[(1-e^2)^{-2} - (1-e)^{-2} \right]
 \end{aligned}$$

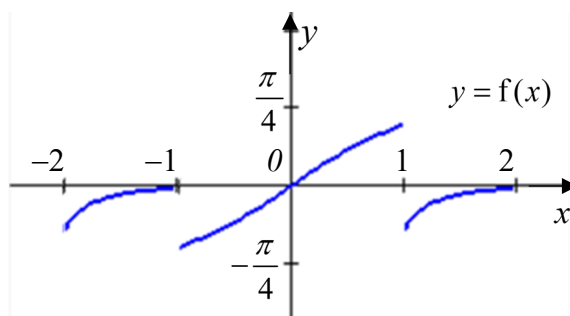
(b)

$$\begin{aligned}
 \int_0^1 \tan^{-1} x \, dx &= \left[x \tan^{-1} x \right]_0^1 - \int_0^1 \frac{x}{1+x^2} dx \\
 &= \left[\tan^{-1} 1 - 0 \right]_0^1 - \frac{1}{2} \int_0^1 \frac{2x}{1+x^2} dx \\
 &= \frac{\pi}{4} - \frac{1}{2} \left[\ln |1+x^2| \right]_0^1 \\
 &= \frac{\pi}{4} - \frac{1}{2} [\ln 2 - \ln 1] \\
 &= \frac{\pi}{4} - \frac{1}{2} \ln 2
 \end{aligned}$$

$$u = \tan^{-1} x \quad \frac{dv}{dx} = 1$$

$$\frac{du}{dx} = \frac{1}{1+x^2} \quad v = x$$

(c)



$$\begin{aligned}\int_{-2}^2 |f(x)| dx &= \int_{-2}^{-1} |f(x)| dx + \int_{-1}^1 |\tan^{-1} x| dx + \int_1^2 \left| \frac{e^x}{(1-e^x)^3} \right| dx \\&= \int_{-1}^1 \left| \frac{e^x}{(1-e^x)^3} \right| dx + \int_{-1}^0 |\tan^{-1} x| dx + \int_0^1 |\tan^{-1} x| dx + \int_1^2 \left| \frac{e^x}{(1-e^x)^3} \right| dx \\&= 2 \int_0^1 \tan^{-1} x dx - 2 \int_1^2 \frac{e^x}{(1-e^x)^3} dx \\&= 2 \underbrace{\left(\frac{\pi}{4} - \frac{1}{2} \ln 2 \right)}_{\text{from part (a)}} - 2 \underbrace{\left[\frac{1}{2} \left((1-e^2)^{-2} - (1-e)^{-2} \right) \right]}_{\text{from part (b)}} \\&= \frac{\pi}{2} - \ln 2 - (1-e^2)^{-2} + (1-e)^{-2}\end{aligned}$$



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(a) Width of each rectangle $= \frac{1}{n}$

When $x = \frac{1}{n}$, $y = \frac{\frac{1}{n}}{\sqrt{1 + \left(\frac{1}{n}\right)^2}}$ (Height of the 1st rectangle)

When $x = \frac{2}{n}$, $y = \frac{\frac{2}{n}}{\sqrt{1 + \left(\frac{2}{n}\right)^2}}$ (Height of the 2nd rectangle)

... ..

When $x = \frac{n-1}{n}$, $y = \frac{\frac{n-1}{n}}{\sqrt{1 + \left(\frac{n-1}{n}\right)^2}}$ (Height of the n^{th} rectangle)

$$\begin{aligned}
 A &= \frac{1}{n} \left(\frac{\frac{1}{n}}{\sqrt{1 + \left(\frac{1}{n}\right)^2}} \right) + \frac{1}{n} \left(\frac{\frac{2}{n}}{\sqrt{1 + \left(\frac{2}{n}\right)^2}} \right) + \frac{1}{n} \left(\frac{\frac{3}{n}}{\sqrt{1 + \left(\frac{3}{n}\right)^2}} \right) + \dots + \frac{1}{n} \left(\frac{\frac{n-1}{n}}{\sqrt{1 + \left(\frac{n-1}{n}\right)^2}} \right) \\
 &= \frac{1}{n^2} \left(\frac{1}{\sqrt{1 + \left(\frac{1}{n}\right)^2}} + \frac{2}{\sqrt{1 + \left(\frac{2}{n}\right)^2}} + \frac{3}{\sqrt{1 + \left(\frac{3}{n}\right)^2}} + \dots + \frac{n-1}{\sqrt{1 + \left(\frac{n-1}{n}\right)^2}} \right) \\
 &= \frac{1}{n^2} \left(\frac{1}{\sqrt{\frac{n^2 + 1^2}{n^2}}} + \frac{2}{\sqrt{\frac{n^2 + 2^2}{n^2}}} + \frac{3}{\sqrt{\frac{n^2 + 3^2}{n^2}}} + \dots + \frac{n-1}{\sqrt{\frac{n^2 + (n-1)^2}{n^2}}} \right) \\
 &= \frac{1}{n^2} \left(\frac{n}{\sqrt{n^2 + 1^2}} + \frac{2n}{\sqrt{n^2 + 2^2}} + \frac{3n}{\sqrt{n^2 + 3^2}} + \dots + \frac{(n-1)n}{\sqrt{n^2 + (n-1)^2}} \right) \\
 &= \frac{1}{n} \left[\frac{1}{\sqrt{n^2 + 1^2}} + \frac{2}{\sqrt{n^2 + 2^2}} + \frac{3}{\sqrt{n^2 + 3^2}} + \dots + \frac{(n-1)}{\sqrt{n^2 + (n-1)^2}} \right] \\
 &= \frac{1}{n} \sum_{r=1}^{n-1} \frac{r}{\sqrt{n^2 + r^2}} \quad (\text{shown})
 \end{aligned}$$

(b)

$$\begin{aligned}
 \lim_{n \rightarrow \infty} A &= \int_0^1 \frac{x}{\sqrt{1+x^2}} dx \\
 &= \frac{1}{2} \int_0^1 2x \cdot (1+x^2)^{-\frac{1}{2}} dx \\
 &= \frac{1}{2} \left[\frac{(1+x^2)^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^1 \\
 &= \left[\sqrt{1+x^2} \right]_0^1 \\
 &= \sqrt{2} - 1
 \end{aligned}$$

(i) Method 1: Integration by Parts

$$\begin{aligned}
 \int_0^1 \frac{x}{\sqrt{x+1}} dx &= \left[2x\sqrt{x+1} \right]_0^1 - \int_0^1 2\sqrt{x+1} dx \\
 &= \left[2\sqrt{2} - 0 \right] - 2 \int_0^1 (x+1)^{\frac{1}{2}} dx \\
 &= 2\sqrt{2} - 2 \left[\frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 \\
 &= 2\sqrt{2} - \frac{4}{3} \left[(x+1)^{\frac{3}{2}} \right]_0^1 \\
 &= 2\sqrt{2} - \frac{4}{3} \left[2^{\frac{3}{2}} - 1^{\frac{3}{2}} \right] \\
 &= 2\sqrt{2} - \frac{4}{3} (2\sqrt{2} - 1) \\
 &= \frac{4}{3} - \frac{2}{3} \sqrt{2} \\
 &= \frac{2}{3} (2 - \sqrt{2}) \quad (\text{shown})
 \end{aligned}$$

$$\begin{array}{lcl}
 u = x & \searrow & \frac{dv}{dx} = \frac{1}{\sqrt{x+1}} = (x+1)^{-\frac{1}{2}} \\
 \frac{du}{dx} = 1 & \swarrow & v = 2\sqrt{x+1}
 \end{array}$$

Method 2: Algebraic Manipulation

$$\begin{aligned}
 \int_0^1 \frac{x}{\sqrt{x+1}} dx &= \int_0^1 \frac{x+1-1}{\sqrt{x+1}} dx \\
 &= \int_0^1 \frac{(x+1)}{\sqrt{x+1}} - \frac{1}{\sqrt{x+1}} dx \\
 &= \int_0^1 \sqrt{x+1} - \frac{1}{\sqrt{x+1}} dx \\
 &= \int_0^1 (x+1)^{\frac{1}{2}} - (x+1)^{-\frac{1}{2}} dx \\
 &= \left[\frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+1)^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^1 \\
 &= \left[\frac{2}{3} (x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} \right]_0^1 \\
 &= \left[\frac{2}{3} (2)^{\frac{3}{2}} - 2(2)^{\frac{1}{2}} \right] - \left[\frac{2}{3} (1)^{\frac{3}{2}} - 2(1)^{\frac{1}{2}} \right] \\
 &= \frac{2}{3} (2\sqrt{2}) - 2\sqrt{2} + \frac{4}{3} \\
 &= \frac{4}{3} - \frac{2}{3} \sqrt{2} \\
 &= \frac{2}{3} (2 - \sqrt{2}) \quad (\text{shown})
 \end{aligned}$$

(ii) Width of each rectangle $= \frac{1}{n}$

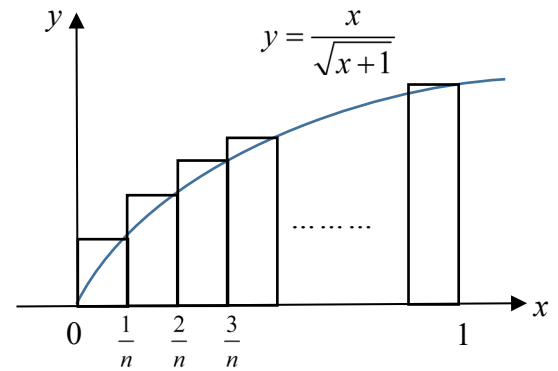
When $x = \frac{1}{n}$, $y = \frac{\frac{1}{n}}{\sqrt{\frac{1}{n}+1}}$ (Height of the 1st rectangle)

When $x = \frac{2}{n}$, $y = \frac{\frac{2}{n}}{\sqrt{\frac{2}{n}+1}}$ (Height of the 2nd rectangle)

When $x = \frac{3}{n}$, $y = \frac{\frac{3}{n}}{\sqrt{\frac{3}{n}+1}}$ (Height of the 3rd rectangle)

... ..

When $x = \frac{n}{n}$, $y = \frac{\frac{n}{n}}{\sqrt{\frac{n}{n}+1}}$ (Height of the n^{th} rectangle)

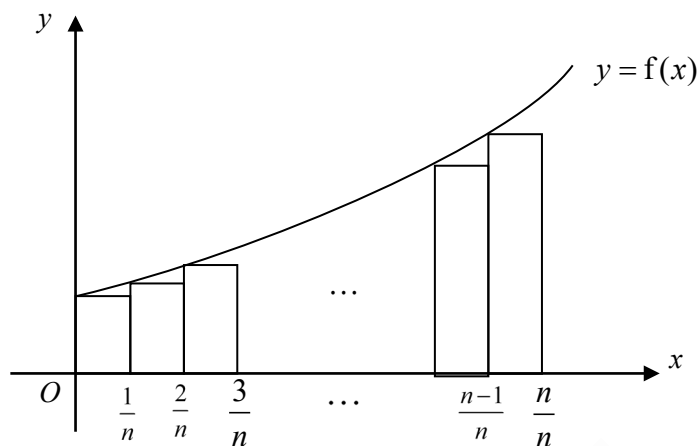


$$\begin{aligned}
 A &= \frac{1}{n} \left(\frac{\frac{1}{n}}{\sqrt{\frac{1}{n}+1}} \right) + \frac{1}{n} \left(\frac{\frac{2}{n}}{\sqrt{\frac{2}{n}+1}} \right) + \left(\frac{\frac{3}{n}}{\sqrt{\frac{3}{n}+1}} \right) \dots + \frac{1}{n} \left(\frac{\frac{n}{n}}{\sqrt{\frac{n}{n}+1}} \right) \\
 &= \frac{1}{n^2} \left(\frac{1}{\sqrt{\frac{1+n}{n}}} + \frac{2}{\sqrt{\frac{2+n}{n}}} + \frac{3}{\sqrt{\frac{3+n}{n}}} + \dots + \frac{n}{\sqrt{\frac{n+n}{n}}} \right) \\
 &= \frac{1}{n^2} \left(\frac{\sqrt{n}}{\sqrt{1+n}} + \frac{2\sqrt{n}}{\sqrt{2+n}} + \frac{3\sqrt{n}}{\sqrt{3+n}} + \dots + \frac{n\sqrt{n}}{\sqrt{n+n}} \right) \\
 &= \frac{\sqrt{n}}{n^2} \cdot \frac{\sqrt{n}}{\sqrt{n}} \left(\frac{1}{\sqrt{1+n}} + \frac{2}{\sqrt{2+n}} + \frac{3}{\sqrt{3+n}} + \dots + \frac{n}{\sqrt{2n}} \right) \\
 &= \frac{1}{n\sqrt{n}} \left(\frac{1}{\sqrt{1+n}} + \frac{2}{\sqrt{2+n}} + \frac{3}{\sqrt{3+n}} + \dots + \frac{n}{\sqrt{2n}} \right)
 \end{aligned}$$

All the rectangles above the curve enclosed the region bounded by the curve and the x -axis and include a small portion above the curve. Thus, the total area of all the rectangles is more than the area

bounded by the curve and x -axis. Hence, $\frac{1}{n\sqrt{n}} \left[\frac{1}{\sqrt{n+1}} + \frac{2}{\sqrt{n+2}} + \frac{3}{\sqrt{n+3}} + \dots + \frac{n}{\sqrt{2n}} \right] > \frac{2}{3}(2-\sqrt{2})$.

(a)

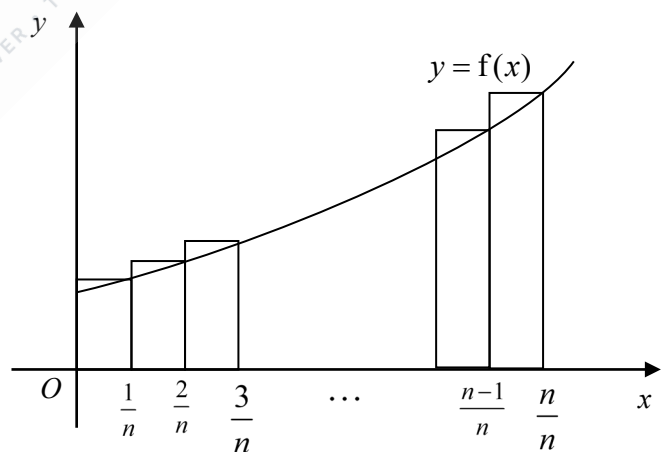


Area of n rectangles under the above curve between $x = 0$ and $x = 1$,

$$\begin{aligned}
 &= \frac{1}{n}f(0) + \frac{1}{n}f\left(\frac{1}{n}\right) + \frac{1}{n}f\left(\frac{2}{n}\right) + \dots + \frac{1}{n}f\left(\frac{n-1}{n}\right) \\
 &= \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)
 \end{aligned}$$

Since area of n rectangles is an **underestimate of / less than** the actual area under curve $= \int_0^1 f(x) \, dx$,

(b) $\frac{1}{n} \sum_{r=1}^n f\left(\frac{r}{n}\right)$



(c)(i) Given that $f(x) = x^2 + 1$ and $n = 10$.

$$\begin{aligned}\text{Lower bound of area of } A &= \frac{1}{10} \sum_{r=0}^9 f\left(\frac{r}{10}\right) \\ &= \frac{1}{10} \sum_{r=0}^9 \left(\left(\frac{r}{10}\right)^2 + 1 \right) \\ &= 1.285\end{aligned}$$

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$$\frac{1}{10} \sum_{x=0}^9 \left(\left(\frac{x}{10} \right)^2 + 1 \right)$$

Ans ▸ Dec 1.285

$$\begin{aligned}\text{Upper bound of area of } A &= \frac{1}{10} \sum_{r=1}^{10} f\left(\frac{r}{10}\right) \\ &= \frac{1}{10} \sum_{r=1}^{10} \left(\left(\frac{r}{10}\right)^2 + 1 \right) \\ &= 1.385\end{aligned}$$

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$$\frac{1}{10} \sum_{x=1}^{10} \left(\left(\frac{x}{10} \right)^2 + 1 \right)$$

Ans ▸ Dec 1.385

(c)(ii) From GC : $\int_0^1 x^2 + 1 \, dx = 1.3333$

Difference from $\int_0^1 x^2 + 1 \, dx$

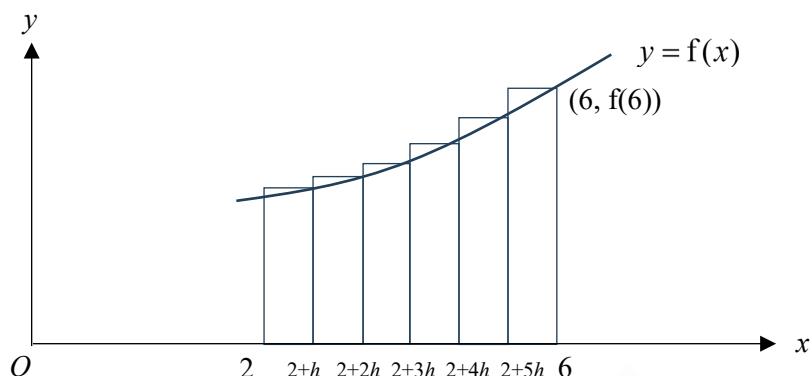
$$(a) \text{ lower limit} = \frac{1.3333 - 1.285}{1.3333} = \frac{0.0483}{1.3333} = 3.62\%$$

$$(b) \text{ upper limit} = \frac{1.385 - 1.3333}{1.3333} = \frac{0.0517}{1.3333} = 3.88\%$$

The lower limit is a better estimate since it has a smaller % / absolute difference from the exact value.

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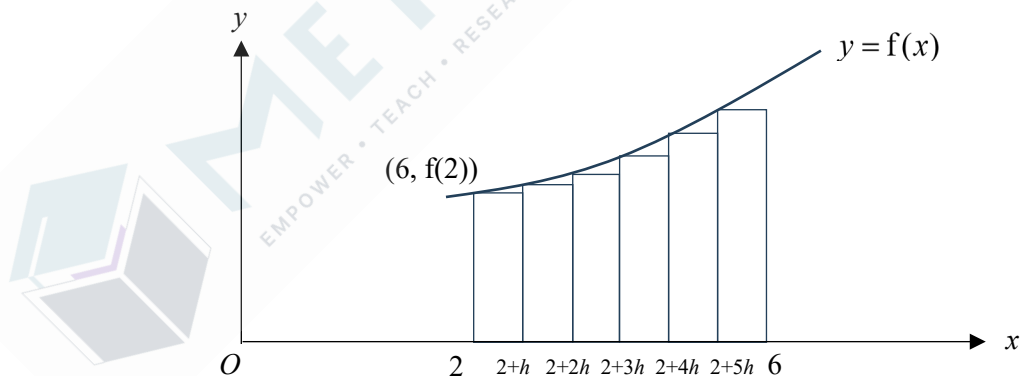
(i) $h = \frac{6-2}{6} = \frac{2}{3}$



$$\sum_{n=1}^6 (f(2+nh))h = f(2+h)h + f(2+2h)h + f(2+3h)h + f(2+4h)h + f(2+5h)h + f(2+6h)h$$

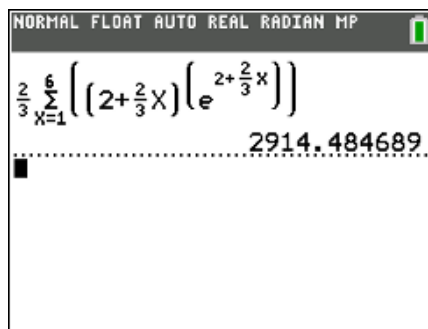
represents the sum of areas of six rectangles drawn and the rectangles are above the curve, so it is greater than area of A .

(ii) $\sum_{n=0}^5 (f(2+nh))h$



(iii) When $h = \frac{2}{3}$, $f(x) = xe^x$

$$\begin{aligned} \text{Upper bound of area of } A &= \sum_{n=1}^6 (f(2+nh))h \\ &= \sum_{n=1}^6 \left[\left(2 + \frac{2}{3}n \right) e^{\left(2 + \frac{2}{3}n \right)} \right] \frac{2}{3} \\ &= \frac{2}{3} \sum_{n=1}^6 \left[\left(2 + \frac{2}{3}n \right) e^{\left(2 + \frac{2}{3}n \right)} \right] \\ &\approx 2914 \end{aligned}$$

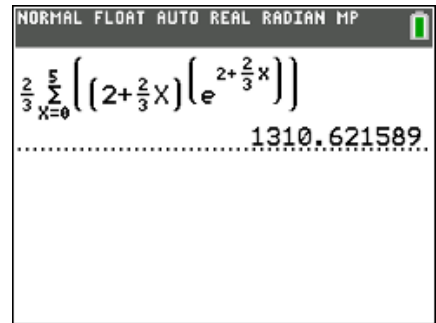


Lower bound of area of $A = \sum_{n=0}^5 (f(2+nh))h$

$$= \sum_{n=0}^5 \left[\left(2 + \frac{2}{3}n \right) e^{\left(2 + \frac{2}{3}n \right)} \right] \frac{2}{3}$$

$$= \frac{2}{3} \sum_{n=0}^5 \left[\left(2 + \frac{2}{3}n \right) e^{\left(2 + \frac{2}{3}n \right)} \right]$$

$$\approx 1311$$



(iv)

Exact area of A

$$= \int_2^6 x e^x \, dx$$

$$= \left[x e^x \right]_2^6 - \int_2^6 e^x \, dx$$

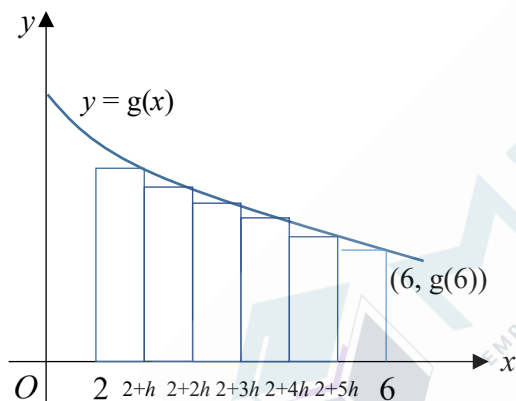
$$= 6e^6 - 2e^2 - \left[e^x \right]_2^6$$

$$= 6e^6 - 2e^2 - \left[e^6 - e^2 \right]$$

$$= 5e^6 - e^2$$

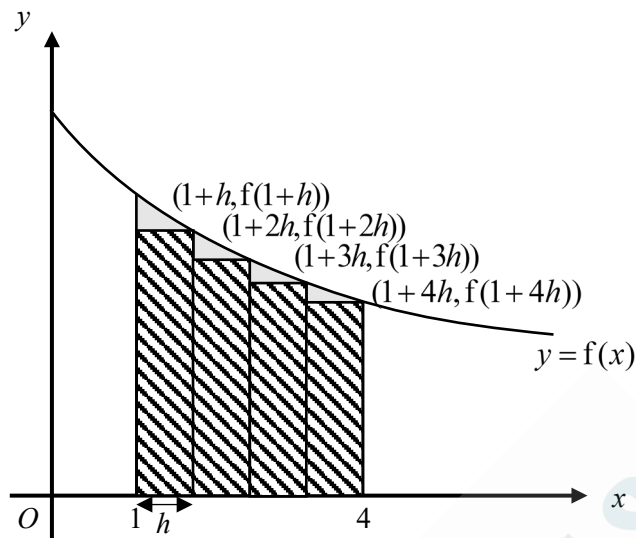
$$\begin{array}{l} u = x \quad \swarrow \quad \frac{dv}{dx} = e^x \\ \frac{du}{dx} = 1 \quad \longleftarrow \quad v = e^x \end{array}$$

(v)



19

(a) $h = \frac{4-1}{4} = \frac{3}{4}$



From the diagram,

Total area of shaded rectangles < Area under the curve from
 $x = 1$ to $x = 4$

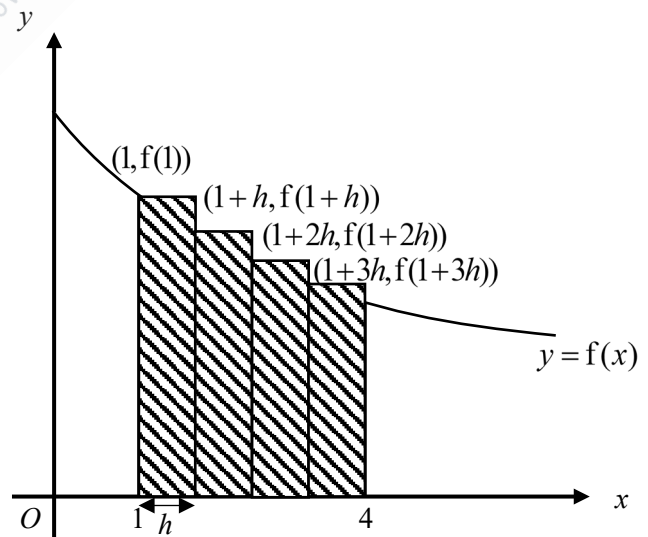
$$\therefore \sum_{n=1}^4 [f(1+nh)]h < \text{Area of } A$$

(b) $\sum_{n=0}^3 [f(1+nh)]h$

or $\sum_{n=1}^4 [f(0.25+nh)]h$

or $\sum_{n=1}^4 [f(4-nh)]h$

or $\sum_{n=1}^4 [f(1+(n-1)h)]h$



(c) When $h = \frac{3}{4}$, $f(x) = -2\ln(x+1) + 7$

$$\begin{aligned}\text{Lower bound of area of } A &= \sum_{n=1}^4 (f(1+nh))h \\ &= \sum_{n=1}^4 \left[-2\ln\left(1 + \frac{3}{4}n + 1\right) + 7 \right] \frac{3}{4} \\ &= \frac{3}{4} \sum_{n=1}^4 \left[-2\ln\left(2 + \frac{3}{4}n\right) + 7 \right] \\ &\approx 13.0\end{aligned}$$

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$$\frac{3}{4} \sum_{x=1}^4 \left(-2\ln\left(2 + \frac{3}{4}x\right) + 7 \right)$$

13.01891884

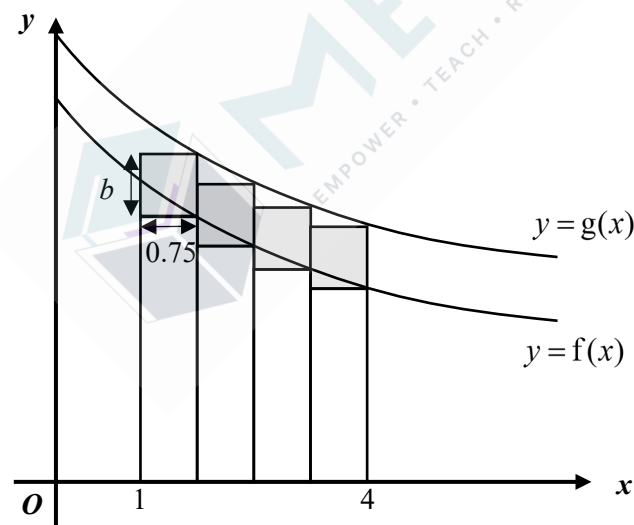
$$\begin{aligned}\text{Upper bound of area of } A &= \sum_{n=0}^3 (f(1+nh))h \\ &= \sum_{n=0}^3 \left[-2\ln\left(1 + \frac{3}{4}n + 1\right) + 7 \right] \frac{3}{4} \\ &= \frac{3}{4} \sum_{n=0}^3 \left[-2\ln\left(2 + \frac{3}{4}n\right) + 7 \right] \\ &\approx 14.4\end{aligned}$$

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$$\frac{3}{4} \sum_{x=0}^3 \left(-2\ln\left(2 + \frac{3}{4}x\right) + 7 \right)$$

14.39335493

(d) **Method 1:**



Since translation by b units in the positive y -direction will result in an increase in the area by 4 rectangles (represented by c) with each length 0.75 units and breadth b units,

$$\begin{aligned}\therefore c &= 4 \times 0.75b \\ c &= 3b\end{aligned}$$

Method 2:

$$y = f(x)$$

↓ translate b units in the positive y -direction

$$y = f(x) + b = g(x)$$

$$\text{Given } \sum_{n=1}^4 [g(1+nh)]h = \sum_{n=1}^4 [f(1+nh)]h + c$$

$$\begin{aligned}\sum_{n=1}^4 [g(1+nh)]h &= \sum_{n=1}^4 [f(1+nh) + b]h \\ &= \sum_{n=1}^4 [f(1+nh)]h + \sum_{n=1}^4 bh \\ &= \sum_{n=1}^4 [f(1+nh)]h + 4bh\end{aligned}$$

$$\therefore c = 4b(0.75)$$

$$c = 3b$$

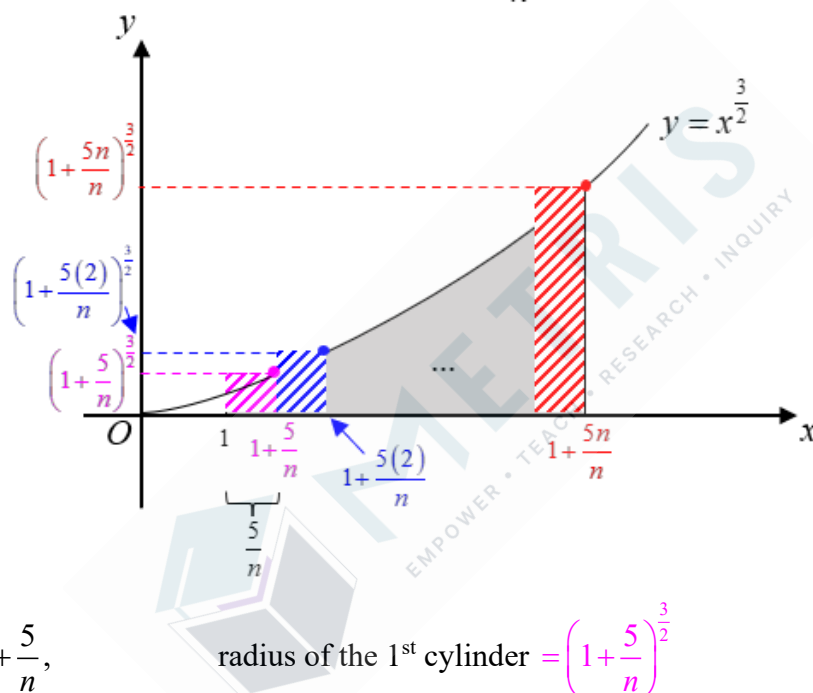


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(a)

$$\begin{aligned}
 \pi \int_1^6 x^3 \, dx &= \pi \left[\frac{x^4}{4} \right]_1^6 \\
 &= \frac{\pi}{4} (6^4 - 1^4) \\
 &= \frac{1295}{4} \pi
 \end{aligned}$$

(b) The thickness of each circular disc is obtained by dividing x values from 1 to 6 into n equal parts, forming n discs of equal thickness of $\frac{5}{n}$ (which is the height of each cylinder)



When $x = 1 + \frac{5}{n}$,

radius of the 1st cylinder $= \left(1 + \frac{5}{n}\right)^{\frac{3}{2}}$

When $x = 1 + \frac{5(2)}{n}$,

radius of the 2nd cylinder $= \left(1 + \frac{5(2)}{n}\right)^{\frac{3}{2}}$

...

When $x = 1 + \frac{5(n)}{n}$,

radius of the n^{th} cylinder $= \left(1 + \frac{5n}{n}\right)^{\frac{3}{2}}$

$$\begin{aligned}
 V &= \pi \left[\left(1 + \frac{5}{n}\right)^{\frac{3}{2}} \right]^2 \left(\frac{5}{n}\right) + \pi \left[\left(1 + \frac{5(2)}{n}\right)^{\frac{3}{2}} \right]^2 \left(\frac{5}{n}\right) + \dots + \pi \left[\left(1 + \frac{5n}{n}\right)^{\frac{3}{2}} \right]^2 \left(\frac{5}{n}\right) \\
 &= \frac{5\pi}{n} \left[\left(1 + \frac{5}{n}\right)^3 + \left(1 + 2\left(\frac{5}{n}\right)\right)^3 + \dots + \left(1 + n\left(\frac{5}{n}\right)\right)^3 \right] \\
 &= \frac{5\pi}{n} \sum_{r=1}^n \left(1 + r\left(\frac{5}{n}\right)\right)^3 \quad (\text{Shown})
 \end{aligned}$$

(c)

$$\begin{aligned} V &= \frac{5\pi}{n} \sum_{r=1}^n \left(1 + r \left(\frac{5}{n} \right) \right)^3 \\ &= \frac{5\pi}{n} \sum_{r=1}^n \left[1 + 3 \left(\frac{5r}{n} \right) + 3 \left(\frac{5r}{n} \right)^2 + \left(\frac{5r}{n} \right)^3 \right] \\ &= \frac{5\pi}{n} \left\{ \sum_{r=1}^n \left(1 + \frac{15r}{n} \right) + \frac{75}{n^2} \sum_{r=1}^n r^2 + \frac{125}{n^3} \sum_{r=1}^n r^3 \right\} \\ &= \frac{5\pi}{n} \left\{ \underbrace{\sum_{r=1}^n 1}_{\substack{n \text{ copies of } 1}} + \frac{15}{n} \underbrace{\sum_{r=1}^n r}_{\substack{\text{AP: } a=1, \\ \text{no. of terms} = n}} + \frac{75}{n^2} \sum_{r=1}^n r^2 + \frac{125}{n^3} \sum_{r=1}^n r^3 \right\} \\ &= \frac{5\pi}{n} \left\{ n + \frac{15}{n} \left(\frac{n}{2} \right) (1+n) + \frac{75}{n^2} \left(\frac{1}{6} \right) n(n+1)(2n+1) + \frac{125}{n^3} \left(\frac{1}{4} \right) n^2 (n+1)^2 \right\} \\ &= \frac{5\pi}{n} \left\{ n + \frac{15}{2} + \frac{15}{2} n + \frac{25}{2n} (n+1)(2n+1) + \frac{125}{4n} (n+1)^2 \right\} \\ &= \frac{5\pi}{n} \left\{ \left(\frac{17n}{2} + \frac{15}{2} \right) + \frac{25}{2n} (2n^2 + 3n + 1) + \frac{125}{4n} (n^2 + 2n + 1) \right\} \\ &= \frac{5\pi}{n} \left\{ \left(\frac{17n}{2} + \frac{15}{2} \right) + \left(25n + \frac{75}{2} + \frac{25}{2n} \right) + \left(\frac{125}{4} n + \frac{125}{2} + \frac{125}{4n} \right) \right\} \\ &= \frac{5\pi}{n} \left(\frac{259}{4} n + \frac{215}{2} + \frac{175}{4n} \right) \\ &= \frac{5\pi}{4} \left(259 + \frac{430}{n} + \frac{175}{n^2} \right) \\ \therefore a &= 430, \quad b = 175 \end{aligned}$$

(d) As $n \rightarrow \infty$, $\frac{430}{n} \rightarrow 0$, $\frac{175}{n^2} \rightarrow 0 \therefore V \rightarrow \frac{5\pi}{4} (259)$

Limit of $V = \frac{1295\pi}{4}$

Using integration, the volume of revolution of the solid formed when R is rotated through 2π radians about the x -axis is given by $\pi \int_1^6 [f(x)]^2 dx$.

Thus, the volume is $\pi \int_1^6 \left[x^{\frac{3}{2}} \right]^2 dx = \pi \int_1^6 x^3 dx = \frac{1295}{4} \pi$ as found in part (a) which is the same value as

the limit of $V = \frac{1295\pi}{4}$. (Verified)

- (a) Width of shaded rectangle = 1

When $x = 1$, height of shaded rectangle = $\frac{1}{1+k}$

$$\text{Area of rectangle} = \frac{1}{1+k} \times 1 = \frac{1}{1+k}$$

$$\text{Area under graph} = \int_{k-1}^k \frac{1}{1+x} dx = [\ln(1+x)]_{k-1}^k = \ln(1+k) - \ln k$$

From the diagram, since area of rectangle is less than area under graph from $k-1$ to k ,

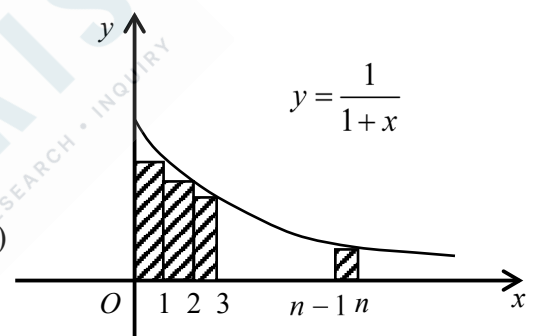
$$\frac{1}{1+k} < \ln(1+k) - \ln k.$$

Area of n rectangles < actual area under the graph from $x = 0$ to $x = n$

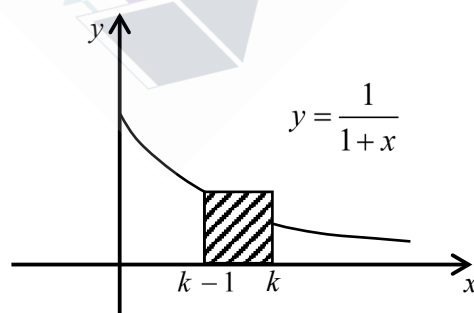
$$\sum_{r=1}^n \frac{1}{r+1} < \int_0^n \frac{1}{1+x} dx$$

$$\underbrace{\frac{1}{2}}_{\text{area of 1st rectangle}} + \underbrace{\frac{1}{3}}_{\text{area of 2nd rectangle}} + \underbrace{\frac{1}{4}}_{\text{area of 3rd rectangle}} + \dots + \underbrace{\frac{1}{1+n}}_{\text{area of nth rectangle}} < [\ln|1+x|]_0^n$$

$$\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{1+n} < \ln(1+n) \quad (\text{deduced})$$



- (b) From diagram,



When $x = k-1$, height of rectangle = $\frac{1}{1+k-1} = \frac{1}{k}$

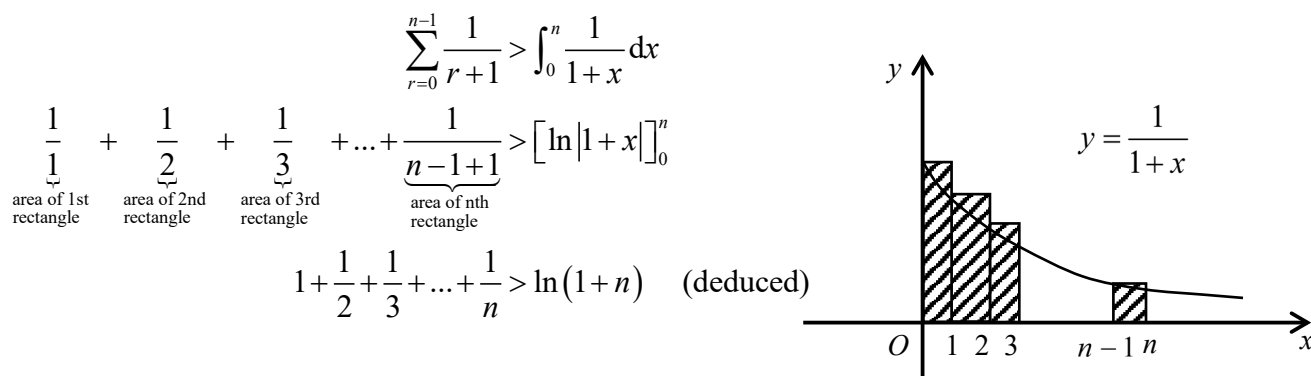
Width of rectangle = 1

$$\text{Area of rectangle} = \frac{1}{k}$$

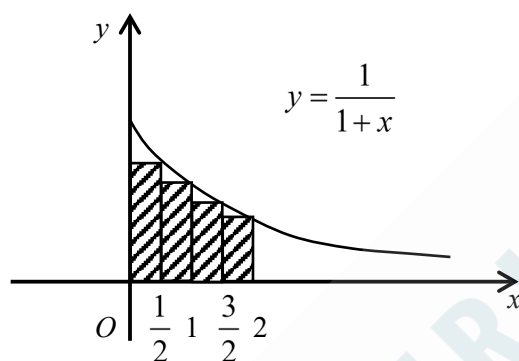
Since area of rectangle is more than area under graph from $k-1$ to k ,

$$\frac{1}{k} > \ln(1+k) - \ln k \quad (\text{shown}).$$

Area of n rectangles $>$ actual area under the graph from $x = 0$ to $x = n$



(c)



Width of each rectangle $= \frac{2-0}{4} = \frac{1}{2}$ (height of the cylinder)

To obtain underestimate, we use the x values of 0.5, 1, 1.5 and 2 to obtain the height of rectangle.

Height of the 4 rectangles are $\frac{1}{1+0.5}, \frac{1}{1+1}, \frac{1}{1+1.5}$ and $\frac{1}{1+2}$ (which are the radii of the cylinder when rotated about the x -axis)

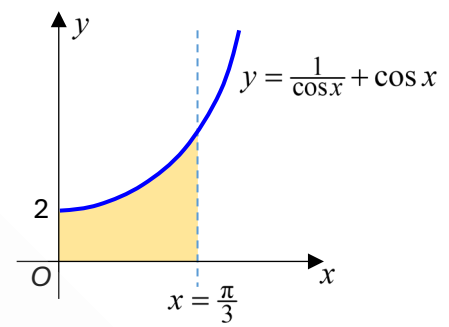
Using the formula for volume of cylinder, the volume of the 4 cylinders that will lead to under-estimation

$$= \pi \left(\frac{1}{1+0.5} \right)^2 (0.5) + \pi \left(\frac{1}{1+1} \right)^2 (0.5) + \pi \left(\frac{1}{1+1.5} \right)^2 (0.5) + \pi \left(\frac{1}{1+2} \right)^2 (0.5)$$

$$= 1.5167$$

$$\approx 1.52 \text{ (3 s.f.)}$$

$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\frac{\pi}{3}} \left(\frac{1}{\cos x} + \cos x \right)^2 dx \\
 &= \pi \int_0^{\frac{\pi}{3}} \left(\frac{1}{\cos^2 x} + \cos^2 x + 2 \right) dx \\
 &= \pi \int_0^{\frac{\pi}{3}} \left(\sec^2 x + \frac{1}{2}(\cos 2x + 1) + 2 \right) dx \\
 &= \pi \left[\tan x + \frac{1}{4}(\sin 2x) + \frac{5}{2}x \right]_0^{\frac{\pi}{3}} \\
 &= \pi \left(\tan \frac{\pi}{3} + \frac{1}{4} \left(\sin \frac{2\pi}{3} \right) + \frac{5}{2} \left(\frac{\pi}{3} \right) \right) \\
 &= \pi \left(\sqrt{3} + \frac{5\pi}{6} + \frac{\sqrt{3}}{8} \right) \\
 &= \pi \left(\frac{9\sqrt{3}}{8} + \frac{5\pi}{6} \right) \text{ unit}^3
 \end{aligned}$$



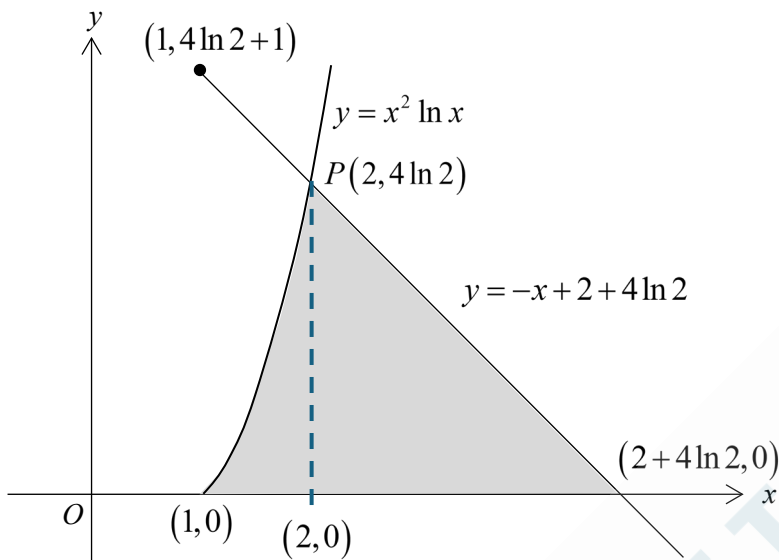
To verify coordinates of P :

Substitute $x = 2$,

$$y = 2^2 \ln 2 = 4 \ln 2$$

$$y = -2 + 2 + 4 \ln 2 = 4 \ln 2$$

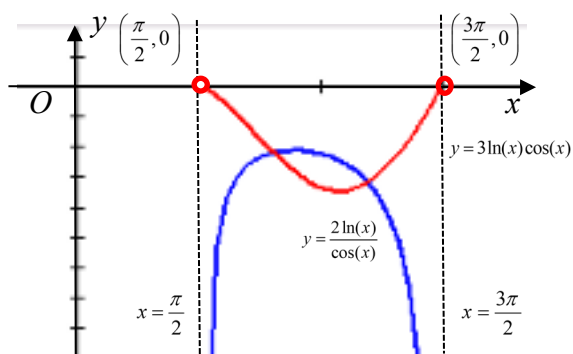
Hence the 2 graphs intersect at $P(2, 4 \ln 2)$.



$$\begin{aligned}
 \text{Area} &= \int_1^2 x^2 \ln x \, dx + \int_2^{2+4 \ln 2} -x + 2 + 4 \ln 2 \, dx \\
 &= \left[\frac{x^3}{3} \ln x \right]_1^2 - \int_1^2 \frac{x^3}{3} \cdot \frac{1}{x} \, dx + \left[-\frac{x^2}{2} + (2 + 4 \ln 2)x \right]_2^{2+4 \ln 2} \\
 &= \frac{8}{3} \ln 2 - \frac{1}{3} \int_1^2 x^2 \, dx + \left[-\frac{(2+4 \ln 2)^2}{2} + (2+4 \ln 2)^2 \right] - \left[-\frac{4}{2} + 2(2+4 \ln 2) \right] \\
 &= \frac{8}{3} \ln 2 - \frac{1}{3} \left[\frac{x^3}{3} \right]_1^2 + \frac{(2+4 \ln 2)^2}{2} - 2 - 8 \ln 2 \\
 &= \frac{8}{3} \ln 2 - \frac{7}{9} + 2(1+2 \ln 2)^2 - 2 - 8 \ln 2 \\
 &= \frac{8}{3} \ln 2 - \frac{7}{9} + 2(1+4 \ln 2 + 4(\ln 2)^2) - 2 - 8 \ln 2 \\
 &= -\frac{7}{9} + \frac{8}{3} \ln 2 + 8(\ln 2)^2 \quad \therefore a = -\frac{7}{9}, b = \frac{8}{3}, c = 8
 \end{aligned}$$

$$\begin{aligned}
 u &= \ln x & \frac{dv}{dx} &= x^2 \\
 \frac{du}{dx} &= \frac{1}{x} & v &= \frac{x^3}{3}
 \end{aligned}$$

(i)



- (ii) From graphing calculator, the two graphs in (i) intersect at $(2.52611, -2.26990)$ (5 d.p.) and $(3.75707, -3.24224)$ (5 d.p.)

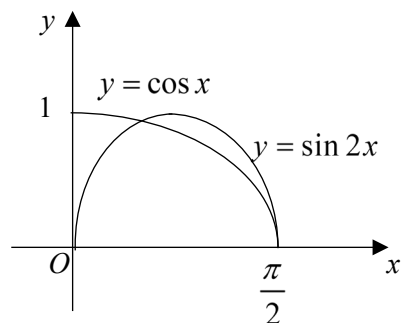
$$\begin{aligned} \text{Area of } R &= \int_{2.52611}^{3.75707} \left[\frac{2 \ln x}{\cos x} - 3 \cos x \ln x \right] dx \\ &\approx 0.94708 \\ &= 0.95 \text{ units}^2. \text{ (2 d.p.)} \end{aligned}$$

(iii)

$$\begin{aligned} \text{Volume of } S &= \pi \int_{2.52611}^{3.75707} \left[(3 \cos x \ln x)^2 - \left(\frac{2 \ln x}{\cos x} \right)^2 \right] dx \\ &\approx 16.968 \\ &= 17.0 \text{ units}^3. \text{ (3 s.f.)} \end{aligned}$$

Volume rotated about x-axis: $\pi \int_{x_1}^{x_2} y^2 dx$

(i)

(ii) $\sin 2x > \cos x$ Consider $\sin 2x = \cos x$

$$\sin 2x - \cos x = 0$$

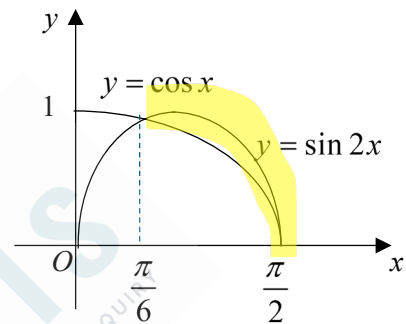
$$2 \sin x \cos x - \cos x = 0$$

$$\cos x(2 \sin x - 1) = 0$$

$$\cos x = 0 \quad \text{or} \quad 2 \sin x - 1 = 0$$

$$x = \frac{\pi}{2} \quad \sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6}$$

From sketch, $\frac{\pi}{6} < x < \frac{\pi}{2}$ 

(iii)

$$\int_0^{\frac{\pi}{2}} |\sin 2x - \cos x| \, dx = -\int_0^{\frac{\pi}{6}} \sin 2x - \cos x \, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin 2x - \cos x \, dx$$

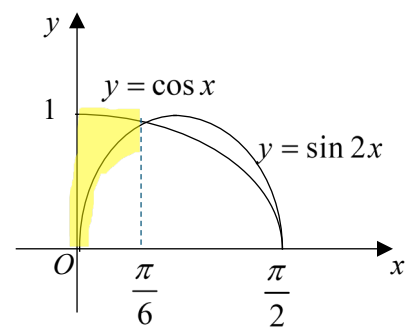
$$= -\left[-\frac{1}{2} \cos 2x - \sin x\right]_0^{\frac{\pi}{6}} + \left[-\frac{1}{2} \cos 2x - \sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= -\left(-\frac{1}{4} - \frac{1}{2} + \frac{1}{2} - 0\right) + \left(\frac{1}{2} - 1 + \frac{1}{4} + \frac{1}{2}\right)$$

When $\frac{\pi}{6} < x < \frac{\pi}{2}$, $\sin 2x > \cos x$
 $\sin 2x - \cos x > 0$
 When $0 < x < \frac{\pi}{6}$, $\sin 2x < \cos x$
 $\sin 2x - \cos x < 0$

(iv)

$$\begin{aligned} \text{Volume} &= \pi \int_0^{\frac{\pi}{2}} \cos^2 x - \sin^2 2x \, dx \\ &= \pi \int_0^{\frac{\pi}{6}} \frac{\cos 2x + 1}{2} - \frac{1 - \cos 4x}{2} \, dx \\ &= \frac{\pi}{2} \int_0^{\frac{\pi}{6}} \cos 2x + 1 - (1 - \cos 4x) \, dx \\ &= \frac{\pi}{2} \int_0^{\frac{\pi}{6}} \cos 2x + \cos 4x \, dx \\ &= \frac{\pi}{2} \left[\frac{\sin 2x}{2} + \frac{\sin 4x}{4} \right]_0^{\frac{\pi}{6}} \\ &= \frac{\pi}{2} \left[\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{8} \right] \\ &= \frac{3\sqrt{3}}{16} \pi \end{aligned}$$



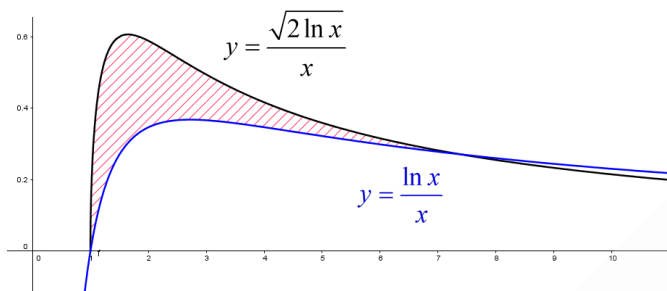
(a)

$$\begin{aligned}\int \frac{(\ln x)^2}{x^2} dx &= -\frac{1}{x}(\ln x)^2 - \int 2\left(\frac{1}{x}\right)\ln x\left(-\frac{1}{x}\right) dx \\ &= -\frac{1}{x}(\ln x)^2 + \int \frac{2\ln x}{x^2} dx\end{aligned}$$

$$\begin{array}{lcl}u = (\ln x)^2 & \searrow & \frac{dv}{dx} = \frac{1}{x^2} \\ \frac{du}{dx} = 2\ln x\left(\frac{1}{x}\right) & \swarrow & v = -\frac{1}{x}\end{array}$$

Therefore, $\int \frac{2\ln x}{x^2} dx - \int \frac{(\ln x)^2}{x^2} dx = \frac{(\ln x)^2}{x} + c$ (shown)

(b)



To find x -coordinates of points of intersection between two graphs:

$$\frac{\sqrt{2\ln x}}{x} = \frac{\ln x}{x}$$

$$\sqrt{2\ln x} = \ln x$$

$$2\ln x = (\ln x)^2$$

$$\ln x(\ln x - 2) = 0$$

$$\ln x = 0 \text{ or } \ln x = 2$$

$$x = 1 \text{ or } x = e^2$$

$$\begin{aligned}\text{Volume} &= \pi \int_1^{e^2} \left(\frac{\sqrt{2\ln x}}{x} \right)^2 dx - \pi \int_1^{e^2} \left(\frac{\ln x}{x} \right)^2 dx \\ &= \pi \int_1^{e^2} \frac{2\ln x}{x^2} - \frac{(\ln x)^2}{x^2} dx \\ &= \pi \left[\frac{(\ln x)^2}{x} \right]_1^{e^2} \\ &= \pi \left[\frac{(\ln e^2)^2}{e^2} - \frac{(\ln 1)^2}{1} \right] \\ &= \frac{4\pi}{e^2}\end{aligned}$$

(a) $x = A(2 - 2x) + B$

$$\therefore A = -\frac{1}{2}, B = 1$$

$$\begin{aligned} \int \frac{x}{\sqrt{3+2x-x^2}} dx &= \int \frac{-\frac{1}{2}(-2x+2)+1}{\sqrt{3+2x-x^2}} dx && \text{Completing the square} \\ &= -\frac{1}{2} \int \frac{-2x+2}{\sqrt{3+2x-x^2}} dx + \int \frac{1}{\sqrt{4-(x-1)^2}} dx \\ &= -\frac{1}{2} \frac{(3+2x-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + \sin^{-1}\left(\frac{x-1}{2}\right) + C \\ &= -\sqrt{3+2x-x^2} + \sin^{-1}\left(\frac{x-1}{2}\right) + C \end{aligned}$$

(b)

$$\begin{aligned} A_y &= 2\pi \int_{t_1}^{t_2} x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= 2\pi \int_0^1 (e^t - t) \sqrt{(e^t - 1)^2 + \left(2e^{\frac{t}{2}}\right)^2} dt \\ &= 2\pi \int_0^1 (e^t - t) \sqrt{e^{2t} - 2e^t + 1 + 4e^t} dt \\ &= 2\pi \int_0^1 (e^t - t) \sqrt{e^{2t} + 2e^t + 1} dt \\ &= 2\pi \int_0^1 (e^t - t) \sqrt{(e^t + 1)^2} dt \\ &= 2\pi \int_0^1 (e^t - t)(e^t + 1) dt \\ &= 2\pi \int_0^1 (e^{2t} - te^t + e^t - t) dt \\ &= 2\pi \left\{ \int_0^1 (e^{2t} + e^t - t) dt - \underbrace{\int_0^1 te^t dt}_{\text{Integration by parts}} \right\} \\ &= 2\pi \left\{ \left[\frac{1}{2}e^{2t} + e^t - \frac{1}{2}t^2 \right]_0^1 \right\} - 2\pi \left\{ [te^t]_0^1 - \int_0^1 e^t dt \right\} \\ &= 2\pi \left\{ \left[\frac{1}{2}e^2 + e - \frac{1}{2} \right] - \left[\frac{1}{2} + 1 - 0 \right] \right\} - 2\pi \left\{ [e^1 - 0] - [e^t]_0^1 \right\} \\ &= 2\pi \left\{ \frac{1}{2}e^2 + e - 2 \right\} - 2\pi \{ e^1 - [e^1 - 1] \} \\ &= 2\pi \left[\frac{1}{2}e^2 + e - 2 - 1 \right] \\ &= 2\pi \left(\frac{1}{2}e^2 + e - 3 \right) \end{aligned}$$

(a)

$$\begin{aligned}
 \int e^{2x} \sin x \, dx &= \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x \, dx \\
 &= \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \left[\frac{1}{2} e^{2x} \cos x - \frac{1}{2} \int -e^{2x} \sin x \, dx \right] \\
 &= \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x - \frac{1}{4} \int e^{2x} \sin x \, dx + C'
 \end{aligned}$$

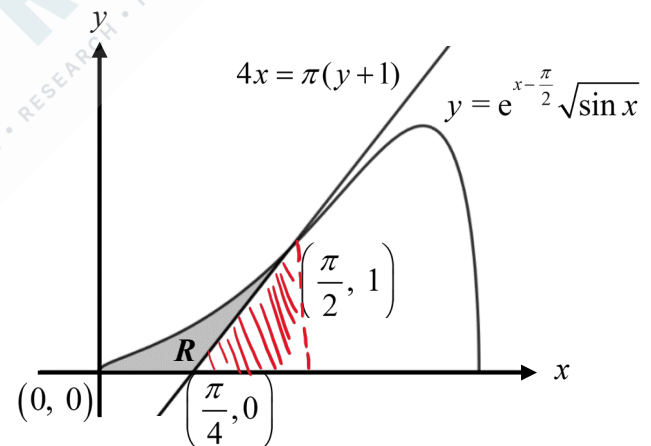
$$\begin{array}{lcl}
 u = \sin x & \searrow & \frac{dv}{dx} = e^{2x} \\
 \frac{du}{dx} = \cos x & \swarrow & v = \frac{e^{2x}}{2}
 \end{array}$$

$$\begin{array}{lcl}
 u = \cos x & \searrow & \frac{dv}{dx} = e^{2x} \\
 \frac{du}{dx} = -\sin x & \swarrow & v = \frac{e^{2x}}{2}
 \end{array}$$

$$\begin{aligned}
 \frac{5}{4} \int e^{2x} \sin x \, dx &= \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x + C' \\
 \int e^{2x} \sin x \, dx &= \frac{4}{5} \left(\frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x + C' \right) \\
 &= \frac{2}{5} e^{2x} \sin x - \frac{1}{5} e^{2x} \cos x + \frac{4}{5} C' \\
 &= \frac{1}{5} e^{2x} (2 \sin x - \cos x) + c, \text{ where } c = \frac{4}{5} C' \quad (\text{shown})
 \end{aligned}$$

(b)

$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\frac{\pi}{2}} \left(e^{x-\frac{\pi}{2}} \sqrt{\sin x} \right)^2 dx - \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{4}{\pi} x - 1 \right)^2 dx \\
 &= \pi \int_0^{\frac{\pi}{2}} e^{2x-\pi} \sin x \, dx - \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{16}{\pi^2} x^2 - \frac{8}{\pi} x + 1 \right) dx \\
 &= \pi \int_0^{\frac{\pi}{2}} e^{-\pi} (e^{2x} \sin x) \, dx - \pi \left[\frac{16}{\pi^2} \cdot \frac{x^3}{3} - \frac{8}{\pi} \cdot \frac{x^2}{2} + x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
 &= \frac{\pi}{e^{\pi}} \int_0^{\frac{\pi}{2}} e^{2x} \sin x \, dx - \pi \left[\left(\frac{16}{3\pi^2} \left(\frac{\pi}{2} \right)^3 - \frac{4}{\pi} \left(\frac{\pi}{2} \right)^2 + \frac{\pi}{2} \right) - \left(\frac{16}{3\pi^2} \left(\frac{\pi}{4} \right)^3 - \frac{4}{\pi} \left(\frac{\pi}{4} \right)^2 + \frac{\pi}{4} \right) \right] \\
 &= \frac{\pi}{e^{\pi}} \left[\frac{1}{5} e^{2x} (2 \sin x - \cos x) \right]_0^{\frac{\pi}{2}} - \pi \left[\left(\frac{2}{3} \pi - \pi + \frac{\pi}{2} \right) - \left(\frac{1}{12} \pi - \frac{\pi}{4} + \frac{\pi}{4} \right) \right] \\
 &= \frac{\pi}{5e^{\pi}} \left[e^{2(\frac{\pi}{2})} \left(2 \sin \frac{\pi}{2} - \cos \frac{\pi}{2} \right) - e^{2(0)} (2 \sin 0 - \cos 0) \right] - \frac{1}{12} \pi^2 \\
 &= \frac{\pi}{5e^{\pi}} [e^{\pi} (2) - (-1)] - \frac{1}{12} \pi^2 \\
 &= \frac{\pi}{5e^{\pi}} [2e^{\pi} + 1] - \frac{1}{12} \pi^2 \\
 &= \frac{\pi}{5} (2 + e^{-\pi}) - \frac{1}{12} \pi^2 \quad \text{where } A = \frac{1}{5}, B = 2, C = -\frac{1}{12}
 \end{aligned}$$



Equation of circle:

$$(x-4)^2 + y^2 = 9$$

$$(x-4)^2 = 9 - y^2$$

$$x-4 = \pm\sqrt{9-y^2}$$

$$x = 4 \pm \sqrt{9-y^2}$$

Since $x < 4$, $x = 4 - \sqrt{9-y^2}$

$$y = -\sqrt{5}x + 3\sqrt{5}$$

$$-\sqrt{5}x = y - 3\sqrt{5}$$

$$x = \frac{y-3\sqrt{5}}{-\sqrt{5}}$$

$$V = \pi \int_0^{\sqrt{5}} \underbrace{\left(\frac{y-3\sqrt{5}}{-\sqrt{5}} \right)^2}_{\text{line}} - \underbrace{\left(4 - \sqrt{9-y^2} \right)^2}_{\text{circle}} dy$$

$$= \pi \int_0^{\sqrt{5}} \frac{(y-3\sqrt{5})^2}{5} - \left[16 - 8\sqrt{9-y^2} + (9-y^2) \right] dy$$

$$= \pi \int_0^{\sqrt{5}} \frac{(y-3\sqrt{5})^2}{5} - (25 - y^2 - 8\sqrt{9-y^2}) dy$$

$$= 31.899 \text{ units}^3 \text{ (3 d.p.)}$$

(a)

$$\begin{aligned}
 \int_0^p \sin^{-1} \frac{t}{3} dt &= \left[t \sin^{-1} \frac{t}{3} \right]_0^p - \int_0^p t \cdot \frac{1}{\sqrt{9-t^2}} dt \\
 &= p \sin^{-1} \frac{p}{3} + \frac{1}{2} \int_0^p -2t(9-t^2)^{-\frac{1}{2}} dt \\
 &= p \sin^{-1} \frac{p}{3} + \frac{1}{2} \left[\frac{\sqrt{9-t^2}}{(\frac{1}{2})} \right]_0^p \\
 &= p \sin^{-1} \frac{p}{3} + \left[\sqrt{9-p^2} - \sqrt{9} \right] \\
 &= p \sin^{-1} \frac{p}{3} + \sqrt{9-p^2} - 3 \quad (\text{Shown})
 \end{aligned}$$

$$\begin{aligned}
 u &= \sin^{-1} \frac{t}{3} & \frac{dv}{dt} &= 1 \\
 \frac{du}{dt} &= \frac{1}{\sqrt{1-\left(\frac{t}{3}\right)^2}} \cdot \frac{1}{3} \\
 &= \frac{1}{3} \frac{1}{\sqrt{\frac{9-t^2}{9}}} \quad \leftarrow v = t \\
 &= \frac{1}{\sqrt{9-t^2}}
 \end{aligned}$$

(b) For the curve $y = x^2$,

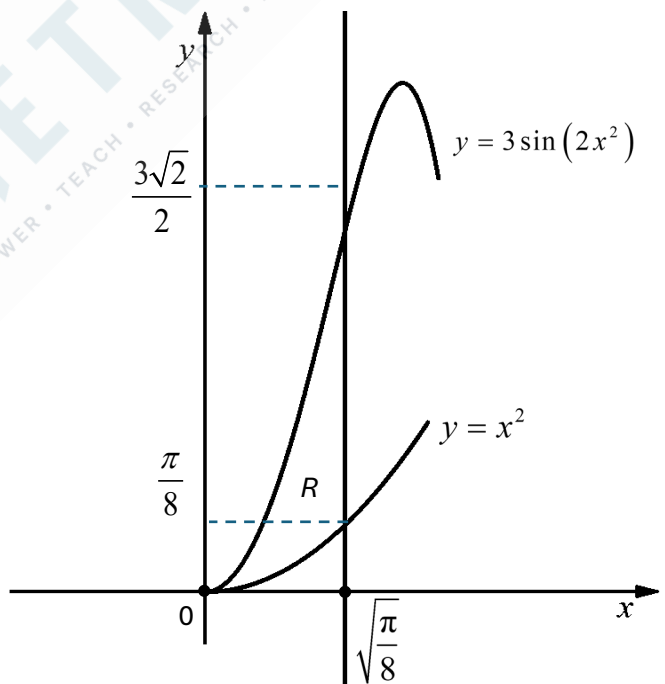
$$\text{When } x = \sqrt{\frac{\pi}{8}}, \quad y = \left(\sqrt{\frac{\pi}{8}} \right)^2 = \frac{\pi}{8}$$

For the curve $y = 3 \sin(2x^2)$,

$$\text{When } x = \sqrt{\frac{\pi}{8}},$$

$$y = 3 \sin \left[2 \left(\sqrt{\frac{\pi}{8}} \right)^2 \right] = 3 \sin \left(\frac{\pi}{4} \right) = \frac{3\sqrt{2}}{2}$$

$$y = 3 \sin(2x^2) \Rightarrow x^2 = \frac{1}{2} \sin^{-1} \left(\frac{y}{3} \right)$$



Required Volume = Volume of cylinder from $y = \frac{\pi}{8}$ to $y = \frac{3\sqrt{2}}{2}$

+ Volume when $y = x^2$ is rotated from $y = 0$ to $y = \frac{\pi}{8}$

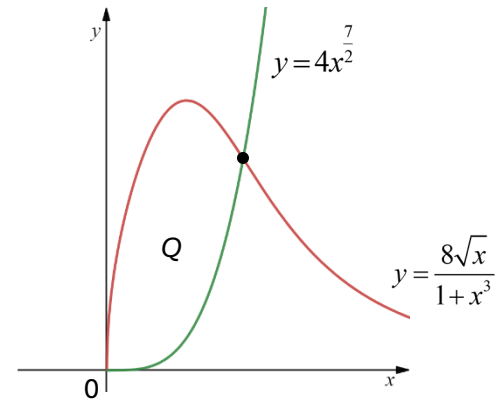
– Volume when $y = 3 \sin(2x^2)$ is rotated from $y = 0$ to $y = \frac{3\sqrt{2}}{2}$

Volume rotated about y-axis: $\pi \int_{y_1}^{y_2} x^2 \, dy$

$$\begin{aligned}
 &= \pi \underbrace{\left(\sqrt{\frac{\pi}{8}}\right)^2}_{\text{radius}} \underbrace{\left(\frac{3\sqrt{2}}{2} - \frac{\pi}{8}\right)}_{\text{height}} + \pi \int_0^{\frac{\pi}{8}} y \, dy - \pi \int_0^{\frac{3\sqrt{2}}{2}} \frac{1}{2} \sin^{-1} \frac{y}{3} \, dy \\
 &= \frac{\pi^2}{8} \left(\frac{3\sqrt{2}}{2} - \frac{\pi}{8}\right) + \pi \left[\frac{y^2}{2}\right]_0^{\frac{\pi}{8}} - \frac{\pi}{2} \int_0^{\frac{3\sqrt{2}}{2}} \sin^{-1} \frac{y}{3} \, dy \\
 &= \frac{\pi^2}{8} \left(\frac{3\sqrt{2}}{2} - \frac{\pi}{8}\right) + \frac{\pi}{2} \left(\frac{\pi}{8}\right)^2 - \frac{\pi}{2} \left[\underbrace{\frac{3\sqrt{2}}{2} \sin^{-1} \left(\frac{\frac{3\sqrt{2}}{2}}{3}\right) + \sqrt{9 - \left(\frac{3\sqrt{2}}{2}\right)^2}}_{\text{from part (a), sub in } p = \frac{3\sqrt{2}}{2}} - 3 \right] \\
 &= \frac{3\sqrt{2}\pi^2}{16} - \frac{\pi^3}{64} + \frac{\pi^3}{128} - \frac{\pi}{2} \left[\frac{3\sqrt{2}}{2} \sin^{-1} \left(\frac{\sqrt{2}}{2}\right) + \sqrt{9 - \frac{9}{2}} - 3 \right] \\
 &= \frac{3\sqrt{2}\pi^2}{16} - \frac{\pi^3}{128} - \frac{\pi}{2} \left[\frac{3\sqrt{2}}{2} \left(\frac{\pi}{4}\right) + 3\sqrt{\frac{1}{2}} - 3 \right] \\
 &= \frac{3\sqrt{2}\pi^2}{16} - \frac{\pi^3}{128} - \frac{3\sqrt{2}\pi^2}{16} - \frac{3\sqrt{2}\pi}{4} + \frac{3\pi}{2} \\
 &= \frac{3\pi}{2} \left(1 - \frac{\sqrt{2}}{2}\right) - \frac{\pi^3}{128} \text{ cubic metres}
 \end{aligned}$$

(c) Using GC, points of intersection: $(0,0)$, $(1,4)$

$$\begin{aligned}\text{Area} &= \int_0^1 \left(\frac{8\sqrt{x}}{1+x^3} - 4x^{\frac{7}{2}} \right) dx \\ &= 3.299901526 \\ &\approx 3.3 \text{ (1 d.p.)}\end{aligned}$$



(d) Volume of main structure $= \frac{3\pi}{2} \left(1 - \frac{\sqrt{2}}{2} \right) - \frac{\pi^3}{128} \text{ m}^3$

Mass of main structure

$$\begin{aligned}&= \left[\frac{3\pi}{2} \left(1 - \frac{\sqrt{2}}{2} \right) - \frac{\pi^3}{128} \right] \times 1463.46 \\ &= 1665.403197 \text{ kg}\end{aligned}$$

Let h m be the thickness of the prism base.

Volume of prism base $= 3.299901526h \text{ m}^3$

Mass of prism structure $= 3.299901526h \times 2550 = 8414.748892h \text{ kg}$

Total mass $= (8414.748892h + 1665.403197) \text{ kg}$

Expected weight per square metre of the monument

$$\begin{aligned}&= \frac{(8414.748892h + 1665.403197) \times \frac{9.81}{1000}}{3.299901526} \\ &= 25.0155h + 4.950937242\end{aligned}$$

$$25.0155h + 4.950937242 \leq 20$$

$$h \leq 0.60159$$

Hence, only if the thickness of the prism base is less than 60.159 cm, then no special foundation will be needed. Should it be between 60.159 cm and 70 cm, a special foundation is needed. Hence, the engineer's claim is not correct.